THE UBIQUITY OF THE SYMPLECTIC HAMILTONIAN EQUATIONS IN MECHANICS

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Abstract. In this paper, we derive a "hamiltonian formalism" for a wide class of mechanical systems, including classical hamiltonian systems, nonholonomic systems, some classes of servomechanism... This construction strongly relies in the geometry characterizing the different systems. In particular, we obtain that the class of the so-called algebroids covers a great variety of mechanical systems. Finally, as the main result, a hamiltonian symplectic realization of systems defined on algebroids is obtained.

1. Introduction

One of the most important equations in Mathematics and Physics are certainly the Hamilton equations:

$$\dot{q}^i = \frac{\partial H}{\partial p_i}, \qquad \dot{p}_i = -\frac{\partial H}{\partial q^i},$$

where (q^i, p_i) are canonical coordinates. Symplectic geometry allows us to write these equations in an intrinsic way (see [1, 24]),

$$i_{X_H}\omega_Q = dH \ . \tag{1.1}$$

In equation (1.1), $H : T^*Q \to \mathbb{R}$ represents a Hamiltonian function defined on the cotangent bundle T^*Q of a configuration manifold Q and ω_Q is the canonical symplectic form of the cotangent bundle (in canonical coordinates, $\omega_Q = dq^i \wedge dp_i$). The skew-symmetry of the canonical symplectic form leads to conservative properties for the Hamiltonian vector field X_H (preservation of the energy). On the contrary, in other type of systems this conservative behavior is not required. For instance, from the symmetry of a riemmanian metric it follows dissipative properties for the gradient vector field (see [11]).

It is an universal belief that Equation (1.1) is only valid for free Hamiltonian systems. For other type of systems, Equation (1.1) is, in general, not longer valid (for instance, in the presence of nonholonomic constraints or dissipative forces, or in the case of gradient systems). In these cases, it is necessary to modify Equation (1.1) adding some extra-terms of different nature: dissipative forces, constraint forces, etc... (see [2, 6, 7, 8, 11, 29]).

Our approximation adopts a new point of view. First, it is necessary to understand the underlying geometry of Equation (1.1) which will permit us to conclude that Hamilton's equations have an ubiquity property: *many different mechanical systems can be described by a symplectic equation*

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constructed in the same way than in the standard case. Of course, this construction relies on the different geometry behind each particular problem. We will show in this paper the main lines of that construction. We should remark that, for the particular construction of Equation (1.1) in each different mechanical problem, it will be necessary to introduce some sophisticated geometric techniques like algebroids, prolongation of structures, lifting of connections [5, 14, 18, 19, 25] among others, which are based on its underlying geometry.

Let us describe our method with more details. If we extract the geometric elements that appear in Equation (1.1) we observe that $\omega_Q = dq^i \wedge dp_i$ is derived from the Liouville 1-form $\lambda_Q = p_i dq^i$, more precisely, $\omega_Q = -d\lambda_Q$ (see [1, 24] for details). In symplectic geometry terms we say that (T^*Q, ω_Q) is an exact symplectic manifold. This structure induces a linear Poisson tensor field Π_{T^*Q} on T^*Q defined by

$$\Pi_{T^*Q}(dF, dG) = \omega_Q(X_F, X_G) \ ,$$

where X_F and X_G are the hamiltonian vector fields corresponding to the functions $F : T^*Q \to \mathbb{R}$ and $G : T^*Q \to \mathbb{R}$, respectively. In canonical coordinates

$$\Pi_{T^*Q} = \frac{\partial}{\partial q^i} \wedge \frac{\partial}{\partial p_i}.$$

A trivial, but interesting, comment is that the classical bracket of vector fields (the standard Lie bracket) is induced by the linear Poisson tensor Π_{T^*Q} (and viceversa). In fact, there exists a one-to-one correspondence between the space of vector fields on Q and the space of linear functions on T^*Q. Indeed, for each vector field $X \in \mathfrak{X}(Q)$ the corresponding linear function $\widehat{X} : T^*Q \to \mathbb{R}$ is given by

$$\widehat{X}(\kappa_q) = \langle \kappa_q, X(q) \rangle,$$

where $\langle \ , \rangle$ is the natural pairing between vectors and covectors, and $\kappa_q \in T_q^*Q$. Therefore, for $X, Y \in \mathfrak{X}(Q)$, the bracket of the two vector fields X and Y is characterized as the unique vector field associated to the linear function $-\Pi_{T^*Q}(d\hat{X}, d\hat{Y})$. Observe that in coordinates

$$\Pi_{T^*Q}(d\hat{X}, d\hat{Y}) = p_j \left(\frac{\partial X^j}{\partial q^i} Y^i - \frac{\partial Y^j}{\partial q^i} X^i \right) = -\widehat{[X, Y]} \ .$$

In a schematic way we have

$$\text{linear Poisson tensor } \Pi_{T^*Q} \quad \longleftrightarrow \quad \text{standard Lie bracket on } Q.$$

That is, we have that there exists a one-to-one correspondence between linear Poisson tensors on T^*Q and Lie algebra structures of vector fields on Q (see [12]).

As a preliminary conclusion, classical hamiltonian formulations strongly relies on the standard Lie bracket of vector fields (or equivalently, the linear Poisson tensor on T^*Q). Modifications of this bracket (or the associated linear tensor) will presumably change the properties of the dynamics. For example, if we do not impose the skew-symmetry of the bracket we will have a dissipative behavior, since, in the cotangent bundle, we will obtain a 2-contravariant linear tensor field which is not necessarily skew-symmetric [11]. Another property that it is possible to drop is the Jacobi identity, which is related with the preservation of the symplectic form by the flow of the hamiltonian vector field. In many interesting cases, as for instance nonholonomic mechanics (see [9, 10, 14]), it is well known that

the Jacobi identity is equivalent to the integrability of the constraints, that is, to holonomic mechanics. Since our objective is to obtain a geometric framework including all these cases, it is necessary to work without imposing Jacobi identity, from the beginning, to our tensor field or associated bracket. Moreover, the role of the tangent bundle is not essential, and we may change it for an arbitrary vector bundle, and then the linear contravariant tensor field will be now defined on its dual bundle E^* (see [38]).

We will show that the category of algebroids is general enough to cover all the cases that we want to analyze. An algebroid (see [15, 18, 19, 21, 31]) is, roughly speaking, a vector bundle $\tau_E : E \to Q$, equipped with a bilinear bracket of sections $B_E : \Gamma(\tau_E) \times \Gamma(\tau_E) \to \Gamma(\tau_E)$ and two vector bundle morphisms $\rho^l_E : E \to TQ$ and $\rho^r_E : E \to TQ$ satisfying a Leibniz-type property (see (2.1)). Observe that properties like skew-symmetry or Jacobi identity are not considered in this category. This general structure is equivalent to give a linear 2-contravariant tensor field Π_{E^*} on its dual bundle E^*. In conclusion, we have that

$$\text{linear contravariant tensor field } \Pi_{E^*} \quad \longleftrightarrow \quad \text{algebroid structure on } E\,.$$

The main objective of this paper is to show that the general construction of the hamiltonian symplectic formalism in classical mechanics remains essentially unchanged starting from the more general framework of algebroids. This result, which is proved in three Theorems (Theorems 4.2, 4.3 and 5.1), constitutes the core of our paper. Additionally, we show how to apply these new techniques to several examples of interest: (generalized)-nonholonomic mechanics, dissipative systems, and gradient systems.

The paper is structured as follows. In Section 2 we define the notion of an algebroid and relate this concept with Hamilton equations for general linear 2-contravariant tensor fields. Moreover, some examples of interest are considered: gradient extension of dynamical systems, nonholonomic mechanics and generalized nonholonomic mechanics. In Section 3, it is introduced the notion of an exact symplectic algebroid, structure that will be necessary to formulate the main result of the paper: the construction of a symplectic formulation of hamiltonian mechanics in the context of algebroids in Sections 4 and 5. Finally, we apply the precedent results to the examples considered in Section 2.

2. Algebroids and Hamiltonian Mechanics

Let $\tau_E : E \to Q$ be a real vector bundle over a manifold Q and $\Gamma(\tau_E)$ be the space of sections of $\tau_E : E \to Q$.

Definition 2.1. *An algebroid structure on E is a $\mathbb{R}$-bilinear bracket*

$$B_E : \Gamma(\tau_E) \times \Gamma(\tau_E) \to \Gamma(\tau_E)$$

together with two vector bundles morphisms $\rho^l_E, \rho^r_E : E \to TQ$ (left and right anchors) such that

$$B_E(f\sigma, f'\sigma') = f\rho^l_E(\sigma)(f')\sigma' - f'\rho^r_E(\sigma')(f)\sigma + ff'B_E(\sigma,\sigma') \tag{2.1}$$

for $f, f' \in C^\infty(Q)$ and $\sigma, \sigma' \in \Gamma(\tau_E)$.

The algebroid structure on $\tau_E : E \to Q$ was defined in [15, 18, 19] and it is called a *Leibniz algebroid* in [21, 31].

If the $\mathbb{R}$-bilinear bracket B_E is skew-symmetric we have a *skew-symmetric algebroid structure* [15] (an *almost Lie algebroid structure* in the terminology of [23] or an almost-Lie structure in the terminology of [32]). In such a case, the left anchor coincides with the right anchor: $\rho^l_E = \rho^r_E$. In the sequel, we will denote the bracket of sections in this skew-symmetric case by $[\![\, ,\,]\!]_E$. On the other hand if the bracket $[\![\, ,\,]\!]_E$ defines a Lie algebra structure on the space $\Gamma(\tau_E)$ then the pair $([\![\, ,\,]\!]_E, \rho_E = \rho^l_E = \rho^r_E)$ is a *Lie algebroid structure* on the vector bundle $\tau_E : E \to Q$ (see [28]).

Another interesting case is when the $\mathbb{R}$-bilinear bracket B_E is symmetric, then we have a *symmetric algebroid structure*. In such a case, $\rho^l_E = -\rho^r_E$.

Now, note that there exists a one-to-one correspondence between the space $\Gamma(\tau_E)$ of sections of the vector bundle $\tau_E : E \to Q$ and the space of linear functions on E^*. In fact, if $\sigma \in \Gamma(\tau_E)$ then the corresponding linear function $\widehat{\sigma}$ on E^* is given by

$$\widehat{\sigma}(\kappa) = \kappa(\sigma(\tau_{E^*}(\kappa))) = \langle \kappa, \sigma(\tau_{E^*}(\kappa))\rangle, \quad \text{for } \kappa \in E^*,$$

where $\tau_{E^*} : E^* \to Q$ is the vector bundle projection.

An algebroid structure $(B_E, \rho^l_E, \rho^r_E)$ on a vector bundle $\tau_E : E \to Q$ induces a linear tensor Π_{E^*} of type (2,0) on E^*. In fact, if $\{\cdot,\cdot\}_{\Pi_{E^*}} : C^\infty(E^*) \times C^\infty(E^*) \to C^\infty(E^*)$ is the induced bracket of functions given by

$$\{\varphi, \psi\}_{\Pi_{E^*}} = \Pi_{E^*}(d\varphi, d\psi), \quad \text{for } \varphi, \psi \in C^\infty(E^*),$$

then we have that

$$\begin{aligned} \{\widehat{\sigma}, \widehat{\sigma'}\}_{\Pi_{E^*}} &= -\widehat{B_E(\sigma,\sigma')}, & \{\widehat{\sigma}, f' \circ \tau_{E^*}\}_{\Pi_{E^*}} &= -\rho^l_E(\sigma)(f') \circ \tau_{E^*} \\ \{f \circ \tau_{E^*}, \widehat{\sigma'}\}_{\Pi_{E^*}} &= \rho^r_E(\sigma')(f) \circ \tau_{E^*}, & \{f \circ \tau_{E^*}, f' \circ \tau_{E^*}\}_{\Pi_{E^*}} &= 0, \end{aligned} \tag{2.2}$$

for $\sigma, \sigma' \in \Gamma(\tau_E)$ and $f, f' \in C^\infty(Q)$ (see [15, 18, 19]).

A curve $\gamma : I \to E$ is ρ^l_E-*admissible* (respectively, ρ^r_E-*admissible*) if $\frac{d}{dt}(\tau_E \circ \gamma) = \rho^l_E \circ \gamma$ (respectively, $\frac{d}{dt}(\tau_E \circ \gamma) = \rho^r_E \circ \gamma$).

In the particular case when E is a skew-symmetric algebroid it follows that Π_{E^*} is a linear 2-vector on E^* (or an *almost Poisson structure* on E^* in the terminology of [23]). If E is a Lie algebroid, the bracket $\{\cdot,\cdot\}_{\Pi_{E^*}}$ satisfies the Jacobi identity and Π_{E^*} is a Poisson structure on E^* (see [12, 23, 27, 38]).

Now let $H : E^* \to \mathbb{R}$ be a *Hamiltonian function* on E^*. Then one may consider the *Hamiltonian vector field* $\mathcal{H}^{\Pi_{E^*}}_H$ of H with respect to Π_{E^*}, that is,

$$\mathcal{H}^{\Pi_{E^*}}_H(F) = -\{H, F\}_{\Pi_{E^*}}, \quad \text{for } F \in C^\infty(E^*).$$

The integral curves of the vector field $\mathcal{H}^{\Pi_{E^*}}_H$ are the solutions of the *Hamilton equations* for H. Next, we will obtain some local expressions.

Suppose that (q^i) are local coordinates on Q and that $\{\sigma_\alpha\}$ is a local basis of the space $\Gamma(\tau_E)$ such that

$$B_E(\sigma_\alpha, \sigma_\beta) = (B_E)^\gamma_{\alpha\beta} e_\gamma, \quad \rho^l_E(\sigma_\alpha) = (\rho^l_E)^i_\alpha \frac{\partial}{\partial q^i}, \quad \rho^r_E(\sigma_\beta) = (\rho^r_E)^j_\beta \frac{\partial}{\partial q^j}. \tag{2.3}$$

The local functions $(B_E)^\gamma_{\alpha\beta}, (\rho^l_E)^i_\alpha$ and $(\rho^r_E)^j_\beta$ are called the *local structure functions* of algebroid $\tau_E : E \to Q$.

Denote by (q^i, p_α) the induced local coordinates on E^*. Then using (2.2) and (2.3), it follows that

$$\Pi_{E^*} = (\rho^r_E)^i_\alpha \frac{\partial}{\partial q^i} \otimes \frac{\partial}{\partial p_\alpha} - (\rho^l_E)^j_\beta \frac{\partial}{\partial p_\beta} \otimes \frac{\partial}{\partial q^j} - (B_E)^\gamma_{\alpha\beta} p_\gamma \frac{\partial}{\partial p_\alpha} \otimes \frac{\partial}{\partial p_\beta}.$$

Therefore, the Hamiltonian vector field of H is given by

$$\mathcal{H}^{\Pi_{E^*}}_H = (\rho^l_E)^i_\alpha \frac{\partial H}{\partial p_\alpha} \frac{\partial}{\partial q^i} - \left((\rho^r_E)^j_\beta \frac{\partial H}{\partial q^j} - (B_E)^\gamma_{\alpha\beta} p_\gamma \frac{\partial H}{\partial p_\alpha} \right) \frac{\partial}{\partial p_\beta}, \tag{2.4}$$

which implies that the local expression of the Hamilton equations is

$$\frac{dq^i}{dt} = (\rho^l_E)^i_\alpha \frac{\partial H}{\partial p_\alpha}, \qquad \frac{dp_\beta}{dt} = -\left\{ (\rho^r_E)^j_\beta \frac{\partial H}{\partial q^j} - (B_E)^\gamma_{\alpha\beta} p_\gamma \frac{\partial H}{\partial p_\alpha} \right\}.$$

Remark 2.2. Working with not in general skew-symmetric or symmetric tensors it is possible to distinguish two different dynamics (one on the left and one on the right). Therefore, we can obtain also a different Hamiltonian vector field $\widetilde{\mathcal{H}}^{\Pi_{E^*}}_H$ of H with respect to Π_{E^*}:

$$\widetilde{\mathcal{H}}^{\Pi_{E^*}}_H(F) = \{F, H\}_{\Pi_{E^*}}, \quad \text{for } F \in C^\infty(E^*).$$

In coordinates,

$$\widetilde{\mathcal{H}}^{\Pi_{E^*}}_H = (\rho^r_E)^i_\alpha \frac{\partial H}{\partial p_\alpha} \frac{\partial}{\partial q^i} - \left((\rho^l_E)^j_\beta \frac{\partial H}{\partial q^j} + (B_E)^\gamma_{\beta\alpha} p_\gamma \frac{\partial H}{\partial p_\alpha} \right) \frac{\partial}{\partial p_\beta}.$$

In the sequel we only consider the vector field $\mathcal{H}^{\Pi_{E^*}}_H$ since the analysis for $\widetilde{\mathcal{H}}^{\Pi_{E^*}}_H$ is similar.

2.1. **First example. A symmetric case: Gradient extension of a dynamical system.** Let Q be an n-dimensional manifold. Let $\mathcal{G}$ be a riemannian metric on Q, i.e, a positive-definite symmetric $(0, 2)$-tensor on Q. Associated to $\mathcal{G}$ we have the associated musical isomorphisms

$$\begin{aligned} &\flat_{\mathcal{G}} : \mathfrak{X}(Q) \longrightarrow \Lambda^1(Q), \qquad \flat_{\mathcal{G}}(X)(Y) = \mathcal{G}(X, Y), \\ &\sharp_{\mathcal{G}}(\mu) = \flat^{-1}_{\mathcal{G}}(\mu) \end{aligned}$$

where $X, Y \in \mathfrak{X}(Q)$ and $\mu \in \Lambda^1(Q)$. In coordinates (q^i) on Q, the metric is expressed as $\mathcal{G} = \mathcal{G}_{ij}(q) dq^i \otimes dq^j$.

Fixed a function $f \in C^\infty(M)$, it is defined the gradient vector field associated to f as $\text{grad}_{\mathcal{G}}(f) = \sharp_{\mathcal{G}}(df)$. In coordinates,

$$\text{grad}_{\mathcal{G}}(f) = \mathcal{G}^{ij} \frac{\partial f}{\partial q^j} \frac{\partial}{\partial q^i}$$

where $(\mathcal{G}^{ij})$ is the inverse matrix of $(\mathcal{G}_{ij})$.

Associated with the metric $\mathcal{G}$ there is an affine connection $\nabla^{\mathcal{G}}$, called the *Levi-Civita connection* determined by:

$$\begin{aligned} [X, Y] &= \nabla^{\mathcal{G}}_X Y - \nabla^{\mathcal{G}}_Y X \text{ (symmetry)} \\ X(\mathcal{G}(Y, Z)) &= \mathcal{G}(\nabla^{\mathcal{G}}_X Y, Z) + \mathcal{G}(Y, \nabla^{\mathcal{G}}_X Z) \text{ (metricity)}, \end{aligned} \tag{2.5}$$

where $X, Y, Z \in \mathfrak{X}(Q)$. Locally, $\nabla^{\mathcal{G}}_{\frac{\partial}{\partial q^j}} \frac{\partial}{\partial q^k} = \Gamma^i_{jk} \frac{\partial}{\partial q^i}$ where the Christoffel symbols Γ^i_{jk} of $\nabla^{\mathcal{G}}$ are given by

$$\Gamma^i_{jk} = \frac{1}{2} \mathcal{G}^{ik'} \left(\frac{\partial \mathcal{G}_{k'j}}{\partial q^k} + \frac{\partial \mathcal{G}_{k'k}}{\partial q^j} - \frac{\partial \mathcal{G}_{jk}}{\partial q^{k'}} \right).$$

Consider now the *symmetric product*:

$$B_{TQ}(X,Y)=\nabla^{\mathcal{G}}_X Y+\nabla^{\mathcal{G}}_Y X \qquad X,Y\in\mathfrak{X}(Q).$$

Locally,

$$B_{TQ}(\frac{\partial}{\partial q^j},\frac{\partial}{\partial q^k})=\left(\Gamma^i_{jk}+\Gamma^i_{kj}\right)\frac{\partial}{\partial q^i}=2\Gamma^i_{jk}\frac{\partial}{\partial q^i}.$$

It is well known that the symmetric product is an element crucial in the study of various aspects of mechanical control systems such us controllability, motion planning, (see for example [3]) and also characterize when a distribution is geodesically invariant [26]. Now define the left and right anchors by, $\rho^l_{TQ}=\mathrm{id}_{TQ}$ and $\rho^r_{TQ}=-\mathrm{id}_{TQ}$. The tangent bundle equipped with $(B_{TQ},\rho^l_{TQ},\rho^r_{TQ})$ is a (symmetric) algebroid.

This structure induces a linear tensor Π_{T^*Q} of type (2,0) on T^*Q. In local coordinates (q^i,p_i) on T^*Q, the bracket relations induced by this tensor field are:

$$\begin{aligned}
\{q^i\,,q^j\}_{\Pi_{T^*Q}}&=0, &\qquad \{q^i\,,p_j\}_{\Pi_{T^*Q}}&=-\delta^i_j\\
\{p_i\,,q^j\}_{\Pi_{T^*Q}}&=-\delta^j_i, &\qquad \{p_i\,,p_j\}_{\Pi_{T^*Q}}&=-2p_k\Gamma^k_{ij}\,.
\end{aligned}$$

Given an arbitrary vector field $X\in\mathfrak{X}(Q)$ one may define the function $H_X:T^*Q\longrightarrow\mathbb{R}$ by $H_X(\kappa)=\langle\kappa,X_q\rangle$, for $\kappa\in T^*_qQ$, that is, $H_X=\widehat{X}$. In coordinates, $H_X(q^i,p_i)=p_iX^i(q)$.

The hamiltonian vector field $\mathcal{H}^{\Pi_{T^*Q}}_{H_X}$ is

$$\mathcal{H}^{\Pi_{T^*Q}}_{H_X}=X^i(q)\frac{\partial}{\partial q^i}+p_k\left(\frac{\partial X^k}{\partial q^j}+2\Gamma^k_{ij}X^i\right)\frac{\partial}{\partial p_j},$$

The equations for its integral curves are:

$$\begin{aligned}
\dot q^i&=X^i(q)\\
\dot p_j&=p_k\left(\frac{\partial X^k}{\partial q^j}+2\Gamma^k_{ij}X^i\right)\,.
\end{aligned}\tag{2.6}$$

These equations are the *gradient extension* of the nonlinear equation $\dot q^i=X^i(q)$ (see [11]).

2.2. **Second example. An antisymmetric case: Nonholonomic Mechanics.** Let $\tau_E:E\to Q$ be a Lie algebroid over a manifold Q and denote by $([\![\cdot,\cdot]\!]_E,\rho_E)$ the Lie algebroid structure on E.

Following [23], a *mechanical system subjected to linear nonholonomic constraints on* E is a pair (L,D), where $L:E\to\mathbb{R}$ is a *Lagrangian function of mechanical type*, that is,

$$L(e)=\frac{1}{2}\mathcal{G}(e,e)-V(\tau_E(e)),\qquad\text{for } e\in E,\tag{2.7}$$

with $\mathcal{G}:E\times_Q E\to\mathbb{R}$ a bundle metric on E and D the total space of a vector subbundle $\tau_D:D\to Q$ of E such that $rank D=m$. The vector subbundle D is said to be the *constraint subbundle*.

Denote by $i_D:D\to E$ the canonical inclusion and consider the orthogonal decomposition $E=D\oplus D^\perp$ and the associated orthogonal projectors $P:E\to D$ and $Q:E\to D^\perp$.

The Levita-Civita connection $\nabla^{\mathcal{G}}:\Gamma(\tau_E)\times\Gamma(\tau_E)\to\Gamma(\tau_E)$ associated to the bundle metric $\mathcal{G}$ is defined in a similar way than in (2.5) (see [10]). It is determined by the formula:

$$\begin{aligned}
2\mathcal{G}(\nabla^{\mathcal{G}}_\sigma\sigma',\sigma'')\;=\;&\rho_E(\sigma)(\mathcal{G}(\sigma',\sigma''))+\rho_E(\sigma')(\mathcal{G}(\sigma,\sigma''))-\rho_E(\sigma'')(\mathcal{G}(\sigma,\sigma'))\\
&+\mathcal{G}(\sigma,[\![\sigma'',\sigma']\!]_E)+\mathcal{G}(\sigma',[\![\sigma'',\sigma]\!]_E)-\mathcal{G}(\sigma'',[\![\sigma',\sigma]\!]_E)
\end{aligned}$$

for $\sigma, \sigma', \sigma'' \in \Gamma(\tau_E)$. The solutions of the nonholonomic problem are the ρ_E-admissible curves $\gamma : I \longrightarrow D$ such that (see [9, 10])

$$\nabla^{\mathcal{G}}_{\gamma(t)}(\gamma(t)) + \mathrm{grad}_{\mathcal{G}} V(\tau_D(\gamma(t))) \in D^{\perp}_{\tau_D(\gamma(t))} \,. \tag{2.8}$$

Here, $\mathrm{grad}_{\mathcal{G}} V$ is the section of $\tau_E : E \to Q$ which is characterized by

$$\mathcal{G}(\mathrm{grad}_{\mathcal{G}} V, \sigma) = \rho_E(\sigma)(V), \quad \text{for } \sigma \in \Gamma(\tau_E).$$

Now, we will derive the equations of motion (2.8) using the general procedure introduced in Section 2. First, we define on the vector bundle $\tau_D : D \longrightarrow Q$ the following *skew-symmetric algebroid structure*:

$$[\![\sigma, \sigma']\!]_D = P([\![i_D \circ \sigma, i_D \circ \sigma')]\!]_E), \quad \rho_D(\sigma) = \rho_E(i_D \circ \sigma), \tag{2.9}$$

for $\sigma, \sigma' \in \Gamma(\tau_D)$.

This skew-symmetric algebroid induces a linear almost-Poisson tensor field Π_{D^*} on the dual bundle D^*. In [23], it is shown that this structure is also induced from the linear Poisson bracket $\{\cdot, \cdot\}_{\Pi_{E^*}}$ on E^*:

$$\{\varphi, \psi\}_{\Pi_{D^*}} = \{\varphi \circ i_D^*, \psi \circ i_D^*\}_{\Pi_{E^*}} \circ P^*, \tag{2.10}$$

for $\varphi, \psi \in C^\infty(D^*)$, where $i_D^* : E^* \to D^*$ and $P^* : D^* \to E^*$ are the dual maps of the monomorphism $i_D : D \to E$ and the projector $P : E \to D$, respectively.

Next, suppose that (q^i) are local coordinates on an open subset U of Q and that $\{\sigma_\alpha\} = \{\sigma_a, \sigma_A\}$ is a basis of sections of the vector bundle $\tau_E^{-1}(U) \to U$ such that $\{\sigma_a\}$ (respectively, $\{\sigma_A\}$) is an orthonormal basis of sections of the vector subbundle $\tau_D^{-1}(U) \to U$ (respectively, $\tau_{D^\perp}^{-1}(U) \to U$). We will denote by $(q^i, v^\alpha) = (q^i, v^a, v^A)$ the corresponding local coordinates on E. Observe, now, that in this coordinates the equations defining D as a vector subbundle of E are:

$$v^A = 0, \quad \text{ for all } A.$$

On the other hand, $[\![\sigma_b, \sigma_c]\!]_D = C^a_{bc}\sigma_a$ and $\rho_D(\sigma_a) = (\rho_D)^i_a \frac{\partial}{\partial q^i} = (\rho_E)^i_a \frac{\partial}{\partial q^i}$, where $C^\gamma_{\alpha\beta}$ and $(\rho_E)^i_\alpha$ are the structure functions of the Lie algebroid $\tau_E : E \longrightarrow Q$ with respect to the local basis $\{\sigma_\alpha\}$. If we denote by (q^i, p_a) the induced local coordinates on D^*, then $\{\ ,\ \}_{\Pi_{D^*}}$ is determined by the following bracket relations:

$$\{q^i, q^j\}_{\Pi_{D^*}} = 0, \qquad \{q^i, p_a\}_{\Pi_{D^*}} = -\{p_a, q^i\}_{\Pi_{D^*}} = (\rho_D)^i_a, \qquad \{p_a, p_b\}_{\Pi_{D^*}} = -p_c C^c_{ab} \,.$$

Now, denote by $\langle\ ,\ \rangle_{\mathcal{G}} : E^* \times_Q E^* \to \mathbb{R}$ the bundle metric on E^* induced by $\mathcal{G}$. Then, define the hamiltonian function $H_{E^*} : E^* \longrightarrow \mathbb{R}$ (*the Hamiltonian energy*) by:

$$H_{E^*}(\kappa_q) = \frac{1}{2}\langle \kappa_q, \kappa_q\rangle_{\mathcal{G}} + V(q), \qquad \kappa_q \in E^*_q$$

Next, we consider *the constrained Hamiltonian function* H_{D^*} on D^* given by

$$H_{D^*} = H_{E^*} \circ P^* : D^* \longrightarrow \mathbb{R}. \tag{2.11}$$

Therefore,

$$\mathcal{H}^{\Pi_{D^*}}_{H_{D^*}} = (\rho_D)^i_a \frac{\partial H_{D^*}}{\partial p_a}\frac{\partial}{\partial q^i} - ((\rho_D)^i_b \frac{\partial H_{D^*}}{\partial q^i} - C^c_{ab} p_c \frac{\partial H_{D^*}}{\partial p_a})\frac{\partial}{\partial p_b}, \tag{2.12}$$

and the Hamilton equations are

$$\frac{dq^i}{dt} = (\rho_D)^i_a \frac{\partial H_{D^*}}{\partial p_a}, \quad \frac{dp_b}{dt} = -((\rho_D)^i_b \frac{\partial H_{D^*}}{\partial q^i} - C^c_{ab} p_c \frac{\partial H_{D^*}}{\partial p_a}).$$

In the induced local coordinates:

$$H_{D^*}(q^i, p_a) = \frac{1}{2} \sum_{a=1}^{m} p_a^2 + V(q^i),$$

and the Hamilton equations are

$$\frac{dq^i}{dt} = (\rho_D)^i_a p_a, \quad \frac{dp_b}{dt} = -\left((\rho_D)^i_b \frac{\partial V}{\partial q^i} - C^c_{ab} p_c p_a \right). \tag{2.13}$$

Remark 2.3. *The Legendre transformation* associated with the Lagrangian function L is just the musical isomorphism $\flat_{\mathcal{G}} : E \to E^*$ induced by the bundle metric $\mathcal{G}$ (for the definition of the Legendre transformation associated with a Lagrangian function $L : E \to \mathbb{R}$, see [25]). *The constrained Legendre transformation* $\mathrm{Leg}_{(L,D)} : D \to D^*$ associated with the nonholonomic system (L, D) is the vector bundle isomorphism induced by the restriction of $\mathcal{G}$ to the vector subbundle $\tau_D : D \to Q$ (see [23]). In other words,

$$\mathrm{Leg}_{(L,D)} = i_D^* \circ \mathrm{Leg}_L \circ i_D = i_D^* \circ \flat_{\mathcal{G}} \circ i_D.$$

Thus, it is easy to prove that

$$\mathrm{Leg}_L(q^i, v^\alpha) = (q^i, v^\alpha), \quad \mathrm{Leg}_{(L,D)}(q^i, v^a) = (q^i, v^a).$$

On the other hand, from (2.8), it follows that a curve

$$\gamma : I \to D, \quad \gamma(t) = (q^i(t), v^a(t))$$

is a solution of the nonholonomic problem if and only if

$$\frac{dq^i}{dt} = (\rho_D)^i_a v^a, \quad \frac{dv^b}{dt} = -\left((\rho_D)^i_b \frac{\partial V}{\partial q^i} - C^c_{ab} v^c v^a \right).$$

Therefore, we deduce that a curve $\gamma : I \to D$ is a solution of the nonholonomic problem if and only if the curve $\mathrm{Leg}_{(L,D)} \circ \gamma : I \to D^*$ is a solution of the Hamilton equations (2.13).

Consequently, Eqs. (2.13) may be considered as the *nonholonomic Hamilton equations* for the constrained system (L, D).

2.3. **Third Example. Mixed cases:**

2.3.1. **Generalized nonholonomic systems on Lie algebroids.**

In this section we will discuss Lagrangian systems on a Lie algebroid $\tau_E : E \to Q$ subjected to generalized nonholonomic constraints (see [2, 6]).

As in the classical nonholonomic case, the *kinematic constraints* are described by a vector subbundle $\tau_D : D \to Q$ of $\tau_E : E \to Q$. Therefore, to determine the dynamics it is only necessary to fix a bundle of reaction forces which vanishes on a vector subbundle $\tilde{D}$ of E. We have that, in general, $D \neq \tilde{D}$. The case $D = \tilde{D}$, classical nonholonomic mechanics, was studied in Subsection 2.2. The case $D \neq \tilde{D}$ appears in many interesting problems, for instance when the restriction is realized by the action of a servo mechanism [29] or for rolling tyres as in [6].

We will call $\tau_{\tilde{D}} : \tilde{D} \to Q$ the *variational subbundle*. It is important to note that $\tilde{D}$ can not be deduced from the kinematic constraints, as the virtual displacements are in classical nonholonomic mechanics.

Now, let $\mathcal{G} : E \times_Q E \to \mathbb{R}$ be a bundle metric on the vector bundle $\tau_E : E \to Q$ and $V : Q \to \mathbb{R} \in C^\infty(Q)$. If $\mathrm{rank} D = \mathrm{rank} \tilde{D}$ and $L : E \to \mathbb{R}$ is the Lagrangian function of mechanical type given by

$$L(e) = \frac{1}{2}\mathcal{G}(e,e) - V(\tau_E(e)), \qquad \text{for } e \in E,$$

then the triple $(L, D, \tilde{D})$ will be called a *mechanical system subject to generalized linear nonholonomic constraints* on E.

We will assume that $\tilde{D}$ satisfies the *compatibility condition*

$$E_q = D_q \oplus \tilde{D}_q^\perp, \qquad \forall\, q \in Q, \tag{2.14}$$

with $\tilde{D}_q^\perp$ the orthogonal complement of $\tilde{D}_q$ in E_q with respect to scalar product $\mathcal{G}_q$. It is obvious that, in particular, this property holds in the classical non-holonomic case $\tilde{D} = D$. In the general case, we have that the equations of motion of such a system are given by $\delta L_{\gamma(t)} \in \tilde{D}^0_{\tau_D(\gamma(t))}$, for a ρ_E-admissible curve $\gamma : I \to D$, or equivalently,

$$\begin{cases} \dfrac{d}{dt}(\tau_E \circ \gamma) = \rho_E^l \circ \gamma, \\ \nabla^{\mathcal{G}}_{\gamma(t)}\gamma(t) + grad_{\mathcal{G}} V(q(t)) \in \tilde{D}^\perp_{q(t)}, \\ \gamma(t) \in D_{q(t)}, \end{cases} \tag{2.15}$$

where $q = \tau_D \circ \gamma$.

Now, suppose that (q^i) are local coordinates on an open subset U of Q and that $\{\sigma_\alpha\} = \{\sigma_a, \sigma_A\}$ is a basis of sections of the vector bundle $\tau_E^{-1}(U) \to U$ such that $\{\sigma_a\}$ (respectively, $\{\sigma_A\}$) is an orthonormal basis of sections of the vector subbundle $\tau_D^{-1}(U) \to U$ (respectively, $\tau_{\tilde{D}^\perp}^{-1}(U) \to U$). In other words, we have a local basis of sections adapted to the decomposition $E = D \oplus \tilde{D}^\perp$. We will denote by $(q^i, v^\alpha) = (q^i, v^a, v^A)$ the corresponding local coordinates on E. The equations defining D as a vector subbundle of E are:

$$v^A = 0, \qquad \text{for all } A.$$

Thus, the equations of motion (2.15) are given by

$$\begin{cases} \dot{v}^c = v^a v^b \widetilde{C}^a_{bc} - (\rho^r_D)^i_c \frac{\partial V}{\partial q^i} \\ \dot{q}^i = (\rho^l_D)^i_a v^a. \end{cases} \tag{2.16}$$

where the functions $\widetilde{C}^c_{ab}$, $(\rho^l_D)^i_b$ and $(\rho^r_D)^i_b$ are properly deduced in Appendix B.

Based on the compatibility condition, it seems natural to consider some decompositions of the original vector bundle E. In particular, we will use

$$E = D \oplus D^\perp \text{ and } E = \tilde{D} \oplus D^\perp$$

with associated projectors:

$$P : D \oplus D^\perp \to D \quad \text{and} \quad \Pi : \tilde{D} \oplus D^\perp \to \tilde{D} \tag{2.17}$$

respectively, and the correspondent inclusions $i_D : D \to E$ and $i_{\tilde{D}} : \tilde{D} \to E$.

Next, denote by $(\llbracket\cdot,\cdot\rrbracket_E, \rho_E)$ the Lie algebroid structure on the vector bundle $\tau_E : E \to Q$. Then the bracket on D given by

$$B_D(\sigma, \sigma') := P(\llbracket i_D(\sigma)\,,\, i_{\widetilde{D}} \circ \Pi(\sigma')\rrbracket_E), \quad \text{for } \sigma, \sigma' \in \Gamma(\tau_D) \tag{2.18}$$

and the anchors maps

$$\rho^l_D \;\; := \;\; \rho_E \circ i_D \tag{2.19}$$

$$\rho^r_D \;\; := \;\; \rho_E \circ i_{\widetilde{D}} \circ \Pi \tag{2.20}$$

define an algebroid structure $(B_D, \rho^l_D, \rho^r_D)$ on the vector bundle $\tau_D : D \to Q$ with local structure functions $\widetilde{C}^c_{ab}$, $(\rho^l_D)^i_a$ and $(\rho^r_D)^i_a$ (see Appendix B). Note that, in general, $\widetilde{C}^c_{ab} \neq -\widetilde{C}^c_{ba}$.

On the other hand, if we denote by (q^i, p_a) the corresponding local coordinates on D^* then the linear bracket $\{\;,\;\}_{\Pi_{D^*}}$ on D^* is determined by the following relations:

$$\{q^i\,, q^j\}_{\Pi_{D^*}} = 0, \qquad \{q^i\,, p_a\}_{\Pi_{D^*}} = (\rho^r_D)^i_a, \qquad \{p_a\,, q^i\}_{\Pi_{D^*}} = -(\rho^l_D)^i_a \qquad \{p_a\,, p_b\}_{\Pi_{D^*}} = -p_c\widetilde{C}^c_{ab}\,.$$

Now, we take the Hamiltonian function $H_{E^*} : E^* \to \mathbb{R}$ (the Hamiltonian energy) defined by

$$H_{E^*}(\kappa_q) = \frac{1}{2} < \kappa_q, \kappa_q >_{\mathcal{G}} + V(q), \quad \text{ for } \kappa_q \in E^*_q$$

where $< \cdot, \cdot >_{\mathcal{G}}$ is the bundle metric on $\tau_{E^*} : E^* \to \mathbb{R}$ induced by $\mathcal{G}$. Then, *the constrained Hamiltonian function* $H_{D^*} : D^* \to \mathbb{R}$ is given by

$$H_{D^*} = H_{E^*} \circ P^*.$$

Thus,

$$\mathcal{H}^{\Pi_{D^*}}_{H_{D^*}} = (\rho^l_D)^i_a \frac{\partial H_{D^*}}{\partial p_a}\frac{\partial}{\partial q^i} - \left((\rho^r_D)^j_b \frac{\partial H_{D^*}}{\partial q^j} - \tilde{C}^c_{ab} p_c \frac{\partial H_{D^*}}{\partial p_a}\right) \frac{\partial}{\partial p_b}, \tag{2.21}$$

which implies that the local expression of the Hamilton equations is

$$\frac{dq^i}{dt} = (\rho^l_D)^i_a \frac{\partial H_{D^*}}{\partial p_a}, \qquad \frac{dp_b}{dt} = -\left\{ (\rho^r_D)^j_b \frac{\partial H_{D^*}}{\partial q^j} - \tilde{C}^c_{ab} p_c \frac{\partial H_{D^*}}{\partial p_a}\right\}.$$

In the induced local coordinates:

$$H_{D^*}(q^i, p_a) = \frac{1}{2}\sum_{a=1}^{m} p_a^2 + V(q^i),$$

and the Hamilton equations are

$$\frac{dq^i}{dt} = \sum_{a=1}^{m} (\rho^l_D)^i_a p_a, \quad \frac{dp_b}{dt} = -\left((\rho^r_D)^i_a \frac{\partial V}{\partial q^i} - \sum_{b=1}^{m} \tilde{C}^c_{ab} p_c p_a \right). \tag{2.22}$$

Remark 2.4. Note that if $\mathrm{Leg}_{(L,D)} : D \to D^*$ is the constrained Legendre transformation, that is,

$$\mathrm{Leg}_{(L,D)} = i^*_D \circ \flat_{\mathcal{G}} \circ i_D$$

then, as the nonholonomic case (see Section 2.2), we deduce that a curve $\gamma : I \to D$ is a solution of the motion equations (2.16) if and only if the curve $\mathrm{Leg}_{(L,D)} \circ \gamma : I \to D^*$ is a solution of the Hamilton equations (2.22). Thus, Eqs. (2.22) may be considered as *the generalized nonholonomic Hamilton equations* for the generalized linear nonholonomic system $(L, D, \tilde{D})$.

2.3.2. **Lagrangian mechanics for modifications of the standard Lie bracket.**

Now, we analyze another example that is of our interest since it has a non skew-symmetric bracket. Consider the case of the standard tangent bundle $\tau_{TQ} : TQ \to Q$, a Lagrangian L of mechanical type

$$L(v) = \frac{1}{2}\mathcal{G}(v, v) - V(\tau_{TQ}(v)), \quad \text{for } v \in TQ,$$

and an arbitrary $(1, 2)$-tensor field T:

$$T : \mathfrak{X}(Q) \times \mathfrak{X}(Q) \to \mathfrak{X}(Q)$$

It is easy to show that if we modify the standard Lie bracket $[\cdot, \cdot]$ on $\mathfrak{X}(Q)$ by

$$B_{TQ}(X, Y) = [X, Y] + T(X, Y)$$

then $(TQ, B_{TQ}, id_{TQ}, id_{TQ})$ is an algebroid.

If we take local coordinates (q^i) and

$$T(\frac{\partial}{\partial q^i}, \frac{\partial}{\partial q^j}) = T_{ij}^k \frac{\partial}{\partial q^k}$$

then

$$B_{TQ}(\frac{\partial}{\partial q^i}, \frac{\partial}{\partial q^j}) = T_{ij}^k \frac{\partial}{\partial q^k}$$

and, therefore, $(B_{TQ})_{ij}^k = T_{ij}^k$. Thus, the linear bracket $\{\cdot, \cdot\}_{\Pi_{T^*Q}}$ on T^*Q is characterized by the following relations

$$\{q^i, q^j\}_{\Pi_{T^*Q}} = 0, \qquad \{q^i, p_j\}_{\Pi_{T^*Q}} = -\{p_i, q^j\}_{\Pi_{T^*Q}} = \delta_i^j, \qquad \{p_i, p_j\}_{\Pi_{D^*}} = -p_k T_{ij}^k \; .$$

In this case, the Hamiltonian function $H_{T^*Q} : T^*Q \to \mathbb{R}$ is given by

$$H_{T^*Q}(\kappa) = \frac{1}{2} < \kappa, \kappa >_{\mathcal{G}} + V(\tau_{T^*Q}(\kappa)), \quad \text{for } \kappa \in T^*Q.$$

Thus, if $\mathcal{G} = \mathcal{G}_{ij} dq^i \otimes dq^j$ it follows that

$$H_{T^*Q}(q^i, p_i) = \frac{1}{2}\mathcal{G}^{ij}(q) p_i p_j + V(q)$$

and then the associated Hamilton equations are

$$\frac{dq^i}{dt} = \mathcal{G}^{ik} p_k, \quad \frac{dp_j}{dt} = -\frac{1}{2}\frac{\partial \mathcal{G}^{ik}}{\partial q^j} p_i p_k - \frac{\partial V}{\partial q^j} - T_{ij}^k \mathcal{G}^{il} p_k p_l. \tag{2.23}$$

Note that $\{H_{T^*Q}, H_{T^*Q}\}_{\Pi_{T^*Q}} = -T_{ij}^k \mathcal{G}^{il}\mathcal{G}^{jm} p_k p_l p_m$ and the dynamics has in general a dissipative behavior. An interesting case, is when the tensor field T is skew-symmetric ($T(X, Y) = -T(Y, X)$ for all $X, Y \in \mathfrak{X}(Q)$) then the hamiltonian H_{T^*Q} is preserved, $dH_{T^*Q}/dt = 0$ along the flow. An important example, when this condition is fulfilled, is the following one. Consider, as above, a riemannian manifold $(Q, \mathcal{G})$ and an arbitrary affine connection ∇. Take the $(1, 2)$ tensor field S which encodes the difference between it and the Levi-Civita connection corresponding to the riemannian metric, that is,

$$\nabla_X Y = \nabla_X^{\mathcal{G}} Y + S(X, Y).$$

This tensor field is called the *contorsion tensor field* (see [34]; and also [4, 8]).

Now, consider as $T(X, Y) = S(X, Y) - S(Y, X)$, and the bracket of vector fields:

$$B_{TQ}(X, Y) = \nabla_X Y - \nabla_Y X = [X, Y] + S(X, Y) - S(Y, X). \tag{2.24}$$

We obtain (2.23) but now the flow preserves the Hamiltonian function. Equations (2.23) are important in the modellization of generalized Chaplygin systems ([4, 8] and references therein), where now the connection ∇ is a metric connection with torsion.

3. Exact symplectic algebroids and Hamiltonian vector fields

In this section we will introduce the notion of an *exact symplectic algebroid* and we will prove that for such an algebroid $\tau_{\bar{E}} : \bar{E} \to \bar{Q}$, if F is a real C^∞-function on the base manifold $\bar{Q}$, then F induces a Hamiltonian vector field on $\bar{Q}$.

First, we will give the definitions of the two differentials of a real C^∞-function F on the base manifold $\bar{Q}$ of an arbitrary vector bundle $\tau_{\bar{E}} : \bar{E} \to \bar{Q}$ with an algebroid structure $(B_{\bar{E}}, \rho^l_{\bar{E}}, \rho^r_{\bar{E}})$. We will also give the definition of the differential of a section of the vector bundle $\tau_{\bar{E}^*} : \bar{E}^* \to \bar{Q}$.

In fact, the *left differential* $d^l_{\bar{E}}F$ *of* F is given by

$$(d^l_{\bar{E}}F)(\sigma) = \rho^l_{\bar{E}}(\sigma)(F), \tag{3.1}$$

and the *right differential* $d^r_{\bar{E}}F$ of F is

$$(d^r_{\bar{E}}F)(\sigma) = \rho^r_{\bar{E}}(\sigma)(F), \tag{3.2}$$

for $\sigma \in \Gamma(\tau_{\bar{E}})$.

On the other hand, if $\kappa : \bar{Q} \to \bar{E}^*$ is a section of the dual vector bundle $\tau_{\bar{E}^*} : \bar{E}^* \to \bar{Q}$ then *the differential of* κ is the section $d^{lr}_{\bar{E}}\kappa$ of the vector bundle $\tau_{\otimes^0_2 \bar{E}^*} : \otimes^0_2 \bar{E}^* \to \bar{Q}$ defined by

$$(d^{lr}_{\bar{E}}\kappa)(\sigma, \sigma') = \rho^l_{\bar{E}}(\sigma)(\kappa(\sigma')) - \rho^r_{\bar{E}}(\sigma')(\kappa(\sigma)) - \kappa(B_{\bar{E}}(\sigma, \sigma')) \tag{3.3}$$

for $\sigma, \sigma' \in \Gamma(\tau_{\bar{E}})$. Note that, from (2.1), it follows that $d^{lr}_{\bar{E}}\kappa \in \Gamma(\tau_{\otimes^0_2 \bar{E}^*})$.

If (q^i) are local coordinates on $\bar{Q}$, $\{\sigma_\alpha\}$ is a local basis of $\Gamma(\tau_{\bar{E}})$, $\{\sigma^\alpha\}$ is the dual basis of $\Gamma(\tau_{\bar{E}^*})$ and $\kappa = \kappa_\gamma \sigma^\gamma$ then

$$d^l_{\bar{E}}F = (\rho^l_{\bar{E}})^i_\alpha \frac{\partial F}{\partial q^i}\sigma^\alpha, \qquad d^r_{\bar{E}}F = (\rho^r_{\bar{E}})^i_\alpha \frac{\partial F}{\partial q^i}\sigma^\alpha$$

$$d^{lr}_{\bar{E}}\kappa = \left\{ (\rho^l_{\bar{E}})^i_\beta \frac{\partial \kappa_\gamma}{\partial q^i} - (\rho^r_{\bar{E}})^i_\gamma \frac{\partial \kappa_\beta}{\partial q^i} + (B_{\bar{E}})^\mu_{\beta\gamma}\kappa_\mu \right\} \sigma^\beta \otimes \sigma^\gamma,$$

where $(B_{\bar{E}})^\mu_{\beta\gamma}, (\rho^l_{\bar{E}})^i_\beta$ and $(\rho^r_{\bar{E}})^i_\gamma$ are the local structure functions of $\tau_{\bar{E}} : \bar{E} \to \bar{Q}$ with respect to the local coordinates (q^i) and the local basis $\{\sigma_\alpha\}$. Furthermore, if $F \in C^\infty(\bar{Q})$ and $\kappa \in \Gamma(\tau_{\bar{E}^*})$, we have that

$$d^{lr}_{\bar{E}}(F\kappa) = F d^{lr}_{\bar{E}}\kappa + d^l_{\bar{E}}F \otimes \kappa - \kappa \otimes d^r_{\bar{E}}F.$$

Definition 3.1. *A vector bundle* $\tau_{\bar{E}} : \bar{E} \to \bar{Q}$ *with an algebroid structure* $(B_{\bar{E}}, \rho^l_{\bar{E}}, \rho^r_{\bar{E}})$ *is said to be exact symplectic if there exists* $\lambda_{\bar{E}} \in \Gamma(\tau_{\bar{E}^*})$ *such that the tensor of type* $(0, 2)$ $\Omega_{\bar{E}}$ *defined by* $\Omega_{\bar{E}} = -d^{lr}_{\bar{E}}\lambda_{\bar{E}} \in \Gamma(\tau_{\otimes^0_2(\bar{E})^*})$ *is skew-symmetric and nondegenerate.*

Thus, if $\tau_{\bar{E}} : \bar{E} \to \bar{Q}$ is an exact symplectic algebroid then the map $\flat^l_{\Omega_{\bar{E}}} : \Gamma(\tau_{\bar{E}}) \to \Gamma(\tau_{\bar{E}^*})$ given by

$$\flat^l_{\Omega_{\bar{E}}}(X) = i_X \Omega_{\bar{E}} = X \lrcorner \Omega_{\bar{E}}, \quad \text{for } X \in \Gamma(\tau_{\bar{E}})$$

is an isomorphism of $C^\infty(\bar{Q})$-modules. Therefore, for a real C^∞-function $\bar{H}$ on $\bar{Q}$ (*a Hamiltonian function*) we may consider the *right Hamiltonian section* $\mathcal{H}_{\bar{H}}^{(\Omega_{\bar{E}},r)}$ of $\bar{H}$ defined by

$$\mathcal{H}_{\bar{H}}^{(\Omega_{\bar{E}},r)} = (\flat^l_{\Omega_{\bar{E}}})^{-1}(d^r_{\bar{E}}\bar{H}) \in \Gamma(\tau_{\bar{E}}) \tag{3.4}$$

and the *left-right Hamiltonian vector field* $\mathcal{H}_{\bar{H}}^{(\Omega_{\bar{E}},lr)}$ of $\bar{H}$ on $\bar{Q}$ given by

$$\mathcal{H}_{\bar{H}}^{(\Omega_{\bar{E}},lr)} = \rho^l_{\bar{E}}\left(\mathcal{H}_{\bar{H}}^{(\Omega_{\bar{E}},r)}\right) \in \mathfrak{X}(\bar{Q}).$$

The integral curves of $\mathcal{H}_{\bar{H}}^{(\Omega_{\bar{E}},lr)}$ are called the solutions of the *Hamilton equations* for $\bar{H}$.

4. An exact symplectic formulation of the Hamiltonian dynamics on an algebroid

In this section, we will propose an exact symplectic formulation of the Hamiltonian dynamics on an algebroid. First, we will see how the bracket of sections of an arbitrary algebroid structure $(B_E, \rho^l_E, \rho^r_E)$ on a vector bundle $\tau_E : E \to Q$ may be written in terms of a suitable ρ^l_E-connection and a ρ^r_E-connection (for the definition of a ρ^l_E-connection and a ρ^r_E-connection, see Appendix A). More precisely, we will prove the following result:

Proposition 4.1. (1) *Let $\tau_E : E \to Q$ be a vector bundle and $\rho^l_E : E \to TQ$, $\rho^r_E : E \to TQ$ two anchor maps. If $D^l : \Gamma(\tau_E) \times \Gamma(\tau_E) \to \Gamma(\tau_E)$ (respectively, $D^r : \Gamma(\tau_E) \times \Gamma(\tau_E) \to \Gamma(\tau_E)$) is a ρ^l_E-connection (respectively, ρ^r_E-connection) on $\tau_E : E \to Q$ then $(B_E, \rho^l_E, \rho^r_E)$ is an algebroid structure on $\tau_E : E \to Q$, where $B_E : \Gamma(\tau_E) \times \Gamma(\tau_E) \to \Gamma(\tau_E)$ is the bracket defined by*

$$B_E(\sigma, \sigma') = D^l_\sigma \sigma' - D^r_{\sigma'}\sigma \quad \textit{for } \sigma, \sigma' \in \Gamma(\tau_E). \tag{4.1}$$

(2) *Let $(B_E, \rho^l_E, \rho^r_E)$ be an algebroid structure on a real vector bundle $\tau_E : E \to Q$. Then, there exists a ρ^l_E-connection D^l (respectively, ρ^r_E-connection D^r) such that*

$$B_E(\sigma, \sigma') = D^l_\sigma \sigma' - D^r_{\sigma'}\sigma \quad \textit{for } \sigma, \sigma' \in \Gamma(\tau_E).$$

Proof.

(1) From (A.1) (in Appendix A) and (4.1), it follows that $(B_E, \rho^l_E, \rho^r_E)$ is an algebroid structure on the vector bundle $\tau_E : E \to Q$.

(2) Suppose that $\tilde{D}^l : \Gamma(\tau_E) \times \Gamma(\tau_E) \to \Gamma(\tau_E)$ (respectively, $\tilde{D}^r : \Gamma(\tau_E) \times \Gamma(\tau_E) \to \Gamma(\tau_E)$) is an arbitrary ρ^l_E-connection (respectively, ρ^r_E-connection) on $\tau_E : E \to Q$. Note that the vector bundle $\tau_E : E \to Q$ admits a ρ^l_E-connection and a ρ^r_E-connection (see Remark A.3 in Appendix A). Then, we may introduce the map $T : \Gamma(\tau_E) \times \Gamma(\tau_E) \to \Gamma(\tau_E)$ given by

$$T(\sigma, \sigma') = B_E(\sigma, \sigma') - \tilde{D}^l_\sigma \sigma' + \tilde{D}^r_{\sigma'}\sigma \quad \text{for } \sigma, \sigma' \in \Gamma(\tau_E).$$

It is easy to check that T is a section of the vector bundle $\tau_{\otimes^0_2 E^*} : \otimes^0_2 E^* \to Q$. Thus,

$$B_E(\sigma, \sigma') = D^l_\sigma \sigma' - D^r_{\sigma'}\sigma \quad \text{for } \sigma, \sigma' \in \Gamma(\tau_E),$$

where D^l and D^r are the ρ^l_E-connection and the ρ^r_E-connection, respectively, defined by

$$D^l_\sigma \sigma' = \tilde{D}^l_\sigma \sigma' \quad D^r_{\sigma'}\sigma = \tilde{D}^r_{\sigma'}\sigma - T(\sigma, \sigma').$$

$\square$

Let $\tau_E : E \to Q$ be an algebroid with structure $(B_E, \rho^l_E, \rho^r_E)$, $H : E^* \to \mathbb{R}$ be a Hamiltonian function on E^* and $\mathcal{H}^{\Pi_{E^*}}_H$ be the corresponding Hamiltonian vector field on E^* (see section 2).

Let us, also, consider the vector bundle $\tau_{\mathcal{T}^E_l E^*} : \mathcal{T}^E_l E^* \to E^*$ over E^* whose fiber at the point $\kappa_q \in E^*_q$ is

$$(\mathcal{T}^E_l E^*)_{\kappa_q} = \{(\sigma_q, \tilde{X}_{\kappa_q}) \in E_q \times T_{\kappa_q} E^* \ / \ \rho^l_E(\sigma_q) = (T_{\kappa_q}\tau_{E^*})(\tilde{X}_{\kappa_q})\}.$$

$\mathcal{T}^E_l E^*$ is called the *the left E-tangent bundle to E^** (see [25, 30] and references therein).

Now, we will prove the two main results of our paper.

Theorem 4.2. *The vector bundle $\tau_{\mathcal{T}^E_l E^*} : \mathcal{T}^E_l E^* \to E^*$ admits an exact symplectic algebroid structure.*

Theorem 4.3. *If $\Omega_{\mathcal{T}^E_l E^*}$ is the exact symplectic structure on the algebroid $\tau_{\mathcal{T}^E_l E^*} : \mathcal{T}^E_l E^* \to E^*$ then the left-right Hamiltonian vector field of a hamiltonian function $H : E^* \to \mathbb{R}$ with respect to $\Omega_{\mathcal{T}^E_l E^*}$ is just $\mathcal{H}^{\Pi_{E^*}}_H$, that is,*

$$\mathcal{H}^{(\Omega_{\mathcal{T}^E_l E^*}, lr)}_H = \mathcal{H}^{\Pi_{E^*}}_H.$$

Proof. [Theorem 4.2]

Using Proposition 4.1, we deduce that there exists a ρ^l_E-connection D^l and a ρ^r_E-connection D^r on $\tau_E : E \to Q$ such that

$$B_E(\sigma, \bar{\sigma}) = D^l_\sigma \bar{\sigma} - D^r_{\bar{\sigma}} \sigma, \qquad \text{for } \sigma, \bar{\sigma} \in \Gamma(\tau_E).$$

Denote by

$$ {}^{D^l}_{\kappa_q}\mathbf{h} : E_q \to T_{\kappa_q} E^*, \qquad\qquad {}^{D^r}_{\kappa_q}\mathbf{h} : E_q \to T_{\kappa_q} E^*$$

the (D^l)-horizontal lift and the (D^r)-horizontal lift, respectively, at the point $\kappa_q \in E^*_q$ (see Appendix A, for a detailed description of horizontal and vertical lifts).

Thus, if $\sigma \in \Gamma(\tau_E)$ we can consider the corresponding (D^l)-horizontal lift $\sigma^{(D^l)\mathbf{h}} \in \mathfrak{X}(E^*)$ (respectively, the corresponding (D^r)-horizontal lift $\sigma^{(D^r)\mathbf{h}} \in \mathfrak{X}(E^*)$) given by

$$\sigma^{(D^l)\mathbf{h}}(\kappa_q) = (\sigma(q))^{D^l\mathbf{h}}_{\kappa_q} \quad \text{and} \quad \sigma^{(D^r)\mathbf{h}}(\kappa_q) = (\sigma(q))^{D^r\mathbf{h}}_{\kappa_q}, \quad \text{for } \kappa_q \in E^*_q.$$

Now, we introduce the D^l-horizontal lift of σ to $\Gamma(\tau_{\mathcal{T}^E_l E^*})$ as the section of $\tau_{\mathcal{T}^E_l E^*} : \mathcal{T}^E_l E^* \to E^*$ defined by

$$\sigma^{(D^l)\mathbf{h}}_l(\kappa_q) = (\sigma(q), \sigma^{(D^l)\mathbf{h}}(\kappa_q)), \qquad \text{for } \kappa_q \in E^*_q.$$

Note that the vector field $\sigma^{(D^l)h}$ is τ_{E^*}-projectable on the vector field $\rho^l_E(\sigma) \in \mathfrak{X}(Q)$ (see Appendix A) and, therefore, $\sigma^{(D^l)\mathbf{h}}_l(\kappa_q) \in (\mathcal{T}^E_l E^*)_{\kappa_q}$. Moreover, from (A.3), it follows that

$$\sigma^{(D^l)\mathbf{h}}_l + \bar{\sigma}^{(D^l)\mathbf{h}}_l = (\sigma + \bar{\sigma})^{(D^l)\mathbf{h}}_l, \quad (f\sigma)^{(D^l)\mathbf{h}}_l = (f \circ \tau_{E^*})\sigma^{(D^l)\mathbf{h}}_l, \tag{4.2}$$

On the other hand, if $\nu \in \Gamma(\tau_{E^*})$ then the vertical lift $\nu^{\mathbf{v}}_l \in \Gamma(\mathcal{T}^E_l E^*)$ of ν to $\mathcal{T}^E_l E^*$ is given by

$$\nu^{\mathbf{v}}_l(\kappa_q) = (0, \nu^{\mathbf{v}}(\kappa_q)), \qquad \text{for } \kappa_q \in E^*_q,$$

where $\nu^{\mathbf{v}} \in \mathfrak{X}(E^*)$ is the standard vertical lift of ν (see Appendix A). In this case, we have also that (see (A.6))

$$\nu^{\mathbf{v}}_l + \bar{\nu}^{\mathbf{v}}_l = (\nu + \bar{\nu})^{\mathbf{v}}_l, \quad (f\nu)^{\mathbf{v}}_l = (f \circ \tau_{E^*})\nu^{\mathbf{v}}_l, \tag{4.3}$$

for $\nu, \bar{\nu} \in \Gamma(\tau_{E^*})$ and $f \in C^\infty(Q)$.

It is clear that if $\{\sigma_\alpha\}$ is a local basis of $\Gamma(\tau_E)$ and $\{\sigma^\alpha\}$ is the dual basis fo $\Gamma(\tau_{E^*})$, then $\{(\sigma_\alpha)_l^{(D^l)\mathbf{h}}, (\sigma^\alpha)_l^{\mathbf{v}}\}$ is a local basis of $\Gamma(\mathcal{T}_l^E E^*)$.

Next, let R be a tensor of type $(1,1)$ on $\tau_E : E \to Q$, that is, $R : \Gamma(\tau_E) \to \Gamma(\tau_E)$ is a $C^\infty(Q)$-linear map. Then, there exists a unique vector field $R^{\mathbf{v}}$ on E^*, *the vertical lift of* R, such that

$$R^{\mathbf{v}}(f \circ \tau_{E^*}) = 0 \quad \text{and} \quad R^{\mathbf{v}}(\widehat{\gamma}) = \widehat{R(\gamma)}$$

for $f \in C^\infty(Q)$ and $\gamma \in \Gamma(\tau_E)$. Here, $\widehat{\ }$ denotes the linear function on E^* induced by a section in $\Gamma(\tau_E)$.

The vertical lift of the tensor R to $\mathcal{T}_l^E E^*$ is denoted by $R_l^{\mathbf{v}}$ and given by

$$R_l^{\mathbf{v}}(\kappa_q) = (0, R^{\mathbf{v}}(\kappa_q)), \qquad \text{for } \kappa_q \in E_q^*.$$

Now, we will define the algebroid structure on $\mathcal{T}_l^E E^*$. First, consider the two anchor maps $\rho^l_{\mathcal{T}_l^E E^*} : \mathcal{T}_l^E E^* \to TE^*$ and $\rho^r_{\mathcal{T}_l^E E^*} : \mathcal{T}_l^E E^* \to TE^*$ given by

$$\begin{aligned} \rho^l_{\mathcal{T}_l^E E^*}(\sigma_q, \tilde{X}_{\kappa_q}) &= \tilde{X}_{\kappa_q}, \\ \rho^r_{\mathcal{T}_l^E E^*}(\sigma_q, \tilde{X}_{\kappa_q}) &= \tilde{X}_{\kappa_q} - (\sigma_q)_{\kappa_q}^{(D^l)\mathbf{h}} + (\sigma_q)_{\kappa_q}^{(D^r)\mathbf{h}}, \end{aligned}$$

for $(\sigma_q, \tilde{X}_{\kappa_q}) \in (\mathcal{T}_l^E E^*)_{\kappa_q}$, with $\kappa_q \in E_q^*$.

Note that

$$\begin{aligned} &\rho^l_{\mathcal{T}_l^E E^*}(\sigma_l^{(D^l)\mathbf{h}}) = \sigma^{(D^l)\mathbf{h}}, && \rho^l_{\mathcal{T}_l^E E^*}(\nu_l^{\mathbf{v}}) = \nu^{\mathbf{v}}, \\ &\rho^r_{\mathcal{T}_l^E E^*}(\sigma_l^{(D^l)\mathbf{h}}) = \sigma^{(D^r)\mathbf{h}}, && \rho^r_{\mathcal{T}_l^E E^*}(\nu_l^{\mathbf{v}}) = \nu^{\mathbf{v}}, \end{aligned} \tag{4.4}$$

for $\sigma \in \Gamma(\tau_E)$ and $\nu \in \Gamma(\tau_{E^*})$.

Next, we define a bracket $B_{\mathcal{T}_l^E E^*}$ on the space $\Gamma(\tau_{\mathcal{T}_l^E E^*})$ by lifting the bracket B_E on $\Gamma(\tau_E)$.

Let R be a tensor of type $(1,3)$ on $\tau_E : E \to Q$, that is, $R : \Gamma(\tau_E) \times \Gamma(\tau_E) \times \Gamma(\tau_E) \to \Gamma(\tau_E)$ is a $C^\infty(Q)$-linear map. Then, using (2.1), (4.2), (4.3) and (A.1), we deduce that there exists a unique bracket $B_{\mathcal{T}_l^E E^*}$ on $\Gamma(\mathcal{T}_l^E E^*)$ such that

$$\begin{aligned} &B_{\mathcal{T}_l^E E^*}(\sigma_l^{(D^l)\mathbf{h}}, \bar{\sigma}_l^{(D^l)\mathbf{h}}) = B_E(\sigma, \bar{\sigma})_l^{(D^l)\mathbf{h}} + R(\sigma, \bar{\sigma}, \cdot)_l^{\mathbf{v}} \\ &B_{\mathcal{T}_l^E E^*}(\sigma_l^{(D^l)\mathbf{h}}, \nu_l^{\mathbf{v}}) = (D^l_\sigma \nu)_l^{\mathbf{v}}, \\ &B_{\mathcal{T}_l^E E^*}(\kappa_l^{\mathbf{v}}, \bar{\sigma}_l^{(D^l)\mathbf{h}}) = -(D^r_{\bar{\sigma}} \kappa)_l^{\mathbf{v}} \\ &B_{\mathcal{T}_l^E E^*}(\kappa_l^{\mathbf{v}}, \nu_l^{\mathbf{v}}) = 0 \end{aligned} \tag{4.5}$$

for $\sigma, \bar{\sigma} \in \Gamma(\tau_E)$ and $\kappa, \nu \in \Gamma(\tau_{E^*})$. Note that $R(\sigma, \bar{\sigma}, \cdot) : \Gamma(\tau_E) \to \Gamma(\tau_E)$ is a tensor of type $(1,1)$ on $\tau_E : E \to Q$. Furthermore, from (4.4) and (4.5), it follows that $(B_{\mathcal{T}_l^E E^*}, \rho^l_{\mathcal{T}_l^E E^*}, \rho^r_{\mathcal{T}_l^E E^*})$ is an algebroid structure on $\mathcal{T}_l^E E^*$.

Now, we will endow the algebroid $(\mathcal{T}_l^E E^*, B_{\mathcal{T}_l^E E^*}, \rho^l_{\mathcal{T}_l^E E^*}, \rho^r_{\mathcal{T}_l^E E^*})$ with an exact symplectic structure. In fact, the dual vector bundle $\tau_{(\mathcal{T}_l^E E^*)^*} : (\mathcal{T}_l^E E^*)^* \to E^*$ admits a canonical section $\lambda_{\mathcal{T}_l^E E^*}$, the *Liouville section*, defined by

$$\lambda_{\mathcal{T}_l^E E^*}(\kappa_q)(\sigma_q, \tilde{X}_{\kappa_q}) = \kappa_q(\sigma_q),$$

for $\kappa_q \in E_q^*$ and $(\sigma_q, \tilde{X}_{\kappa_q}) \in (\mathcal{T}_l^E E^*)_{\kappa_q}$.

Note that

$$\lambda_{\mathcal{T}_l^E E^*}(\sigma_l^{(D^l)\mathbf{h}}) = \widehat{\sigma}, \quad \lambda_{\mathcal{T}_l^E E^*}(\kappa_l^{\mathbf{v}}) = 0 \tag{4.6}$$

for $\sigma \in \Gamma(\tau_E)$ and $\kappa \in \Gamma(\tau_{E^*})$.

Consider the tensor of type $(0,2)$ on $\tau_{\mathcal{T}_l^E E^*} : \mathcal{T}_l^E E^* \to E^*$ given by

$$\Omega_{\mathcal{T}_l^E E^*} = -d^{lr}_{\mathcal{T}_l^E E^*} \left(\lambda_{\mathcal{T}_l^E E^*} \right).$$

From (3.3), (4.4), (4.5), (4.6) and (A.2), we deduce that

$$\begin{aligned}
\Omega_{\mathcal{T}_l^E E^*}(\sigma_l^{(D^l)\mathbf{h}}, \bar{\sigma}_l^{(D^l)\mathbf{h}}) &= -\sigma^{(D^l)\mathbf{h}}(\widehat{\bar{\sigma}}) + (\bar{\sigma})^{(D^r)\mathbf{h}}(\widehat{\sigma}) + \lambda_{\mathcal{T}_l^E E^*}\left(B_E(\sigma,\bar{\sigma})_l^{(D^l)\mathbf{h}} + R(\sigma,\bar{\sigma},\cdot)_l^{\mathbf{v}} \right) \\
&= -\widehat{D^l_\sigma \bar{\sigma}} + \widehat{D^r_{\bar{\sigma}} \sigma} + \widehat{B_E(\sigma,\bar{\sigma})} = 0 \\
\Omega_{\mathcal{T}_l^E E^*}(\sigma_l^{(D^l)\mathbf{h}}, \nu_l^{\mathbf{v}}) &= \nu^{\mathbf{v}}(\widehat{\sigma}) \circ \tau_E + \lambda_{\mathcal{T}_l^E E^*}\left((D^l_\sigma \nu)_l^{\mathbf{v}} \right) = \nu(\sigma) \circ \tau_{E^*} \qquad (4.7)\\
\Omega_{\mathcal{T}_l^E E^*}(\kappa_l^{\mathbf{v}}, \bar{\sigma}_l^{(D^l)\mathbf{h}}) &= -\kappa^{\mathbf{v}}(\widehat{\bar{\sigma}}) \circ \tau_E - \lambda_{\mathcal{T}_l^E E^*}\left((D^r_{\bar{\sigma}} \kappa)_l^{\mathbf{v}} \right) = -\kappa(\bar{\sigma}) \circ \tau_{E^*} \\
\Omega_{\mathcal{T}_l^E E^*}(\kappa_l^{\mathbf{v}}, \nu_l^{\mathbf{v}}) &= 0
\end{aligned}$$

for σ, $\bar{\sigma} \in \Gamma(\tau_E)$ and κ, $\nu \in \Gamma(\tau_{E^*})$. As a consequence, $\Omega_{\mathcal{T}_l^E E^*}$ is a skew-symmetric tensor.

In addition, if $\{\sigma_\alpha\}$ is a local basis of $\Gamma(\tau_E)$, we have that $\{(\sigma_\alpha)_l^{(D^l)\mathbf{h}}, (\sigma^\alpha)_l^{\mathbf{v}}\}$ is a local basis of $\Gamma(\tau_{\mathcal{T}_l^E E^*})$ and

$$\Omega_{\mathcal{T}_l^E E^*} = \left((\sigma_\alpha)_l^{(D^l)\mathbf{h}} \right)^* \wedge \left((\sigma^\alpha)_l^{\mathbf{v}} \right)^* \qquad (4.8)$$

where $\left\{ \left((\sigma_\alpha)_l^{(D^l)\mathbf{h}} \right)^*, \left((\sigma^\alpha)_l^{\mathbf{v}} \right)^* \right\}$ is the dual basis of $\{(\sigma_\alpha)_l^{(D^l)\mathbf{h}}, (\sigma^\alpha)_l^{\mathbf{v}}\}$. Therefore, it is clear that $\Omega_{\mathcal{T}_l^E E^*}$ is nondegenerate.

This ends the proof of our theorem. □

Remark 4.4. In the particular case when E is an skew-symmetric algebroid (that is, the bracket B_E is skew-symmetric), the exact symplectic structure $\Omega_{\mathcal{T}_l^E E^*}$ was considered by Popescu et al [33] in order to develop a symplectic description of the Hamiltonian mechanics on skew-symmetric algebroids. In this case, the exact symplectic structure $\Omega_{\mathcal{T}_l^E E^*}$ does not depend on the chosen connection.

Now, suppose that E is a Lie algebroid with Lie algebroid structure $([\![\cdot,\cdot]\!]_E, \rho_E)$. Then, $(\Gamma(\tau_E), [\![\cdot,]\!]_E)$ is a real Lie algebra and

$$[\![\sigma, f\sigma']\!]_E = f[\![\sigma,\sigma']\!]_E + \rho_E(\sigma)(f)\sigma',$$

for $\sigma, \sigma' \in \Gamma(\tau_E)$ and $f \in C^\infty(Q)$. Moreover, we have that the anchor map $\rho_E^l = \rho_E^r = \rho_E$ is a Lie algebra morphism, i.e. $\rho_E([\![\sigma,\sigma']\!]_E) = [\rho_E(\sigma), \rho_E(\sigma')]$, where $[\cdot,\cdot]$ is the standard bracket of vector fields. In addition, it is well known that $\mathcal{T}_l^E E^* = \mathcal{T}^E E^*$ also admits a Lie algebroid structure (see, for instance, [25]).

Now, we will see that our construction permits to recover this Lie algebroid structure.

To do this, consider an arbitrary bundle metric $\mathcal{G} : E \times_Q E \to \mathbb{R}$ on E and denote by $\nabla^{\mathcal{G}} : \Gamma(E) \times \Gamma(E) \to \Gamma(E)$ the *Levi Civita* connection induced by $\mathcal{G}$. Then, we have that

$$[\![\sigma,\sigma']\!]_E = \nabla^{\mathcal{G}}_\sigma \sigma' - \nabla^{\mathcal{G}}_{\sigma'} \sigma, \quad \text{for } \sigma, \sigma' \in \Gamma(\tau_E). \qquad (4.9)$$

On the other hand *the curvature of the connection* $\nabla^{\mathcal{G}}$ is the tensor field of type (1,3) on $\tau_E : E \to Q$

$$R^{\nabla^{\mathcal{G}}} : \Gamma(\tau_E) \times \Gamma(\tau_E) \times \Gamma(\tau_E) \to \Gamma(\tau_E)$$

defined for each $\sigma, \sigma', \sigma'' \in \Gamma(\tau_E)$ as

$$R^{\nabla^{\mathcal{G}}}(\sigma,\sigma',\sigma'') = \nabla^{\mathcal{G}}_\sigma(\nabla^{\mathcal{G}}_{\sigma'}\sigma'') - \nabla^{\mathcal{G}}_{\sigma'}(\nabla^{\mathcal{G}}_\sigma \sigma'') - \nabla^{\mathcal{G}}_{[\![\sigma,\sigma']\!]_E}\sigma''.$$

Using (A.2) and the fact that ρ_E is a Lie algebra morphism, it is easy to prove that

$$[\sigma^{\nabla^{\mathcal{G}}\mathbf{h}}, (\sigma')^{\nabla^{\mathcal{G}}\mathbf{h}}](f \circ \tau_{E^*}) = (\rho_E [\![\sigma, \sigma']\!]_E)(f) \circ \tau_{E^*}$$

and

$$[\sigma^{\nabla^{\mathcal{G}}\mathbf{h}}, (\sigma')^{\nabla^{\mathcal{G}}\mathbf{h}}](\widehat{\sigma''}) = ([\![\sigma, \sigma']\!]_E)^{\nabla^{\mathcal{G}}\mathbf{h}}(\widehat{\sigma''}) + R^{\nabla^{\mathcal{G}}}(\sigma, \sigma', \cdot)^{\mathbf{v}}(\widehat{\sigma''}),$$

for $\sigma, \sigma', \sigma'' \in \Gamma(\tau_E)$, where $\sigma^{\nabla^{\mathcal{G}}\mathbf{h}}$ denotes the $\nabla^{\mathcal{G}}$-horizontal lift of σ to $\mathfrak{X}(E^*)$. Thus, we conclude that

$$[\sigma^{\nabla^{\mathcal{G}}\mathbf{h}}, (\sigma')^{\nabla^{\mathcal{G}}\mathbf{h}}] = [\![\sigma, \sigma']\!]_E^{\nabla^{\mathcal{G}}\mathbf{h}} + \left(R^{\nabla^{\mathcal{G}}}(\sigma, \sigma', \cdot)\right)^{\mathbf{v}}.$$

By a similar argument, using (A.2) and (A.5), we have that

$$[\sigma^{\nabla^{\mathcal{G}}\mathbf{h}}, \nu^{\mathbf{v}}] = (\nabla^{\mathcal{G}}_\sigma \nu)^{\mathbf{v}} \qquad \text{and} \qquad [\kappa^{\mathbf{v}}, \nu^{\mathbf{v}}] = 0, \quad \text{for } \kappa, \nu \in \Gamma(\tau_{E^*}).$$

Therefore, if we replace in (4.5) the tensor R by the curvature $R^{\nabla^{\mathcal{G}}}$ of the connection $\nabla^{\mathcal{G}}$ then we obtain that

$$\begin{aligned}
&B_{\mathcal{T}^E E^*}(\sigma_l^{\nabla^{\mathcal{G}}\mathbf{h}}, \sigma_l'^{\nabla^{\mathcal{G}}\mathbf{h}}) = ([\![\sigma, \sigma']\!]_E, [\sigma^{\nabla^{\mathcal{G}}\mathbf{h}}, (\sigma')^{\nabla^{\mathcal{G}}\mathbf{h}}]),\\
&B_{\mathcal{T}^E E^*}(\sigma_l^{\nabla^{\mathcal{G}}\mathbf{h}}, \nu_l^{\mathbf{v}}) = (0, [\sigma^{\nabla^{\mathcal{G}}\mathbf{h}}, \nu^{\mathbf{v}}]) = -B_{\mathcal{T}^E E^*}(\nu_l^{\mathbf{v}}, \sigma_l^{\nabla^{\mathcal{G}}\mathbf{h}}),\\
&B_{\mathcal{T}^E E^*}(\kappa_l^{\mathbf{v}}, \nu_l^{\mathbf{v}}) = 0.
\end{aligned}$$

Consequently, if σ (respectively, σ') is a section of $\tau_E : E \to Q$ and X (respectively, X') is a vector field on E^* which is τ_{E^*}-projectable on $\rho_E(\sigma)$ (respectively, $\rho_E(\sigma')$) then (σ, X) and (σ', X') are sections of $\tau_{\mathcal{T}^E E^*} : \mathcal{T}^E E^* \to E^*$ and

$$B_{\mathcal{T}^E E^*}((\sigma, X), (\sigma', X')) = ([\![\sigma, \sigma']\!]_E, [X, X']). \tag{4.10}$$

This implies that $B_{\mathcal{T}^E E^*}$ is the canonical Lie bracket on $\Gamma(\tau_{\mathcal{T}^E E^*})$ (see [25]).

In Section 5 we will use the following properties of the curvature of the connection $\nabla^{\mathcal{G}}$:

$$R^{\nabla^{\mathcal{G}}}(\sigma, \sigma')\sigma'' = -R^{\nabla^{\mathcal{G}}}(\sigma', \sigma)\sigma'' \tag{4.11}$$

and

$$R^{\nabla^{\mathcal{G}}}(\sigma, \sigma')\sigma'' + R^{\nabla^{\mathcal{G}}}(\sigma', \sigma'')\sigma + R^{\nabla^{\mathcal{G}}}(\sigma'', \sigma)\sigma' = 0 \qquad \text{(first Bianchi identity)}. \tag{4.12}$$

Note that (4.12) follows using (4.9) and the fact that $[\![\cdot, \cdot]\!]_E$ satisfies the Jacobi identity.

Remark 4.5. A situation which will be useful in the examples is the case when we start with a vector bundle $\tau_E : E \to Q$ with a skew-symmetric algebroid structure (B_E, ρ_E) such that the anchor map $\rho_E : E \to TQ$ is a skew-symmetric algebroid morphism, that is,

$$\rho_E([\![\sigma, \sigma']\!]_E) = [\rho_E(\sigma), \rho_E(\sigma')].$$

Observe that this condition does not imply that $E \to Q$ is a Lie algebroid as in the previous remark. Under this weaker condition it is still possible to choose the tensor R in such a way the bracket defined in equation (4.5) is again the usual bracket defined in (4.10).

Proof. [Theorem 4.3]

Suppose that (q^i) are local coordinates on Q, $\{\sigma_\alpha\}$ is a local basis of $\Gamma(\tau_E)$ and that $(B_E)^\gamma_{\alpha\beta}$, $(\rho^l_E)^i_\alpha$ and $(\rho^r_E)^i_\alpha$ are the local structure functions of E with respect to the local coordinates (q^i) and to the basis $\{\sigma_\alpha\}$. Then, from Proposition 4.1, it is clear that

$$(B_E)^\gamma_{\alpha\beta} = (D^l)^\gamma_{\alpha\beta} - (D^r)^\gamma_{\beta\alpha}$$

with

$$D^l_{\sigma_\alpha}\sigma_\beta = (D^l)^\mu_{\alpha\beta}\sigma_\mu, \qquad D^r_{\sigma_\alpha}\sigma_\beta = (D^r)^\mu_{\alpha\beta}\sigma_\mu.$$

Moreover, if (q^i, p_α) are the corresponding local coordinates on E^*, we have that (see (A.4) and (A.7))

$$(\sigma_\alpha)^{(D^l)\mathbf{h}} = (\rho^l_E)^i_\alpha \frac{\partial}{\partial q^i} + (D^l)^\gamma_{\alpha\beta} p_\gamma \frac{\partial}{\partial p_\beta}, \qquad (\sigma_\alpha)^{(D^r)\mathbf{h}} = (\rho^r_E)^i_\alpha \frac{\partial}{\partial q^i} + (D^r)^\gamma_{\alpha\beta} p_\gamma \frac{\partial}{\partial p_\beta}, \tag{4.13}$$

$$(\sigma^\alpha)^{\mathbf{v}} = \frac{\partial}{\partial p_\alpha}.$$

Now, let $H : E^* \longrightarrow \mathbb{R}$ be a hamiltonian function. From (3.2), (4.4) and (4.13), it follows that

$$d^r_{\mathcal{T}^E_l E^*} H = \left(\frac{\partial H}{\partial q^i}(\rho^r_E)^i_\alpha + \frac{\partial H}{\partial p_\beta}(D^r)^\gamma_{\alpha\beta} p_\gamma\right)\left((\sigma_\alpha)^{(D^l)\mathbf{h}}_l\right)^* + \frac{\partial H}{\partial p_\alpha}\left((\sigma^\alpha)^{\mathbf{v}}_l\right)^*.$$

Therefore, from (3.4) and (4.8), we obtain that the right Hamiltonian section $\mathcal{H}_H^{(\Omega_{\mathcal{T}^E_l E^*}, r)}$ of H with respect to $\Omega_{\mathcal{T}^E_l E^*}$ is

$$\mathcal{H}_H^{(\Omega_{\mathcal{T}^E_l E^*}, r)} = \frac{\partial H}{\partial p_\alpha}(\sigma_\alpha)^{(D^l)\mathbf{h}}_l - \left(\frac{\partial H}{\partial q^i}(\rho^r_E)^i_\alpha + \frac{\partial H}{\partial p_\beta}(D^r)^\gamma_{\alpha\beta} p_\gamma\right)(\sigma^\alpha)^{\mathbf{v}}_l.$$

Using (4.4) and (4.13), the right Hamiltonian section yields the left-right Hamiltonian vector field of H which is

$$\begin{aligned} \rho^l_{\mathcal{T}^E_l E^*}(\mathcal{H}_H^{(\Omega_{\mathcal{T}^E_l E^*}, r)}) &= \frac{\partial H}{\partial p_\alpha}(\rho^l_E)^i_\alpha \frac{\partial}{\partial q^i} - \left(\frac{\partial H}{\partial q^i}(\rho^r_E)^i_\beta - \frac{\partial H}{\partial p_\alpha}((D^l)^\gamma_{\alpha\beta} - (D^r)^\gamma_{\beta\alpha}) p_\gamma\right)\frac{\partial}{\partial p_\beta}, \\ &= \frac{\partial H}{\partial p_\alpha}(\rho^l_E)^i_\alpha \frac{\partial}{\partial q^i} - \left(\frac{\partial H}{\partial q^i}(\rho^r_E)^i_\beta - \frac{\partial H}{\partial p_\alpha}(B_E)^\gamma_{\alpha\beta} p_\gamma\right)\frac{\partial}{\partial p_\beta}. \end{aligned} \tag{4.14}$$

Consequently, from (2.4) and (4.14), we deduce that

$$\mathcal{H}_H^{(\Omega_{\mathcal{T}^E_l E^*}, lr)} = \rho^l_{\mathcal{T}^E_l E^*}(\mathcal{H}_H^{(\Omega_{\mathcal{T}^E_l E^*}, r)}) = \mathcal{H}_H^{\Pi_{E^*}}.$$

□

Remark 4.6. Theorem 4.3 was proved by Popescu *et al.* [33] for the particular case when E is a skew-symmetric algebroid and by de León *et al.* [23] for the particular case when E is a Lie algebroid.

In order to analyze the closedness of the exact symplectic section $\Omega_{\mathcal{T}^E_l E^*}$ we have to define a differential over tensors of type (0,2) on the algebroid $\mathcal{T}^E_l E^*$. It is clear that it is not possible to induce a direct extension of the differential defined in (3.3). Therefore, the idea is to define a differential over skew-symmetric tensors and another differential over symmetric tensors of type (0,2).

Consider an algebroid structure $(B_{\bar{E}}, \rho^l_{\bar{E}}, \rho^r_{\bar{E}})$ over the vector bundle $\tau_{\bar{E}} : \bar{E} \to \bar{Q}$. Then, it is induced over the vector bundle $\tau_{\bar{E}} : \bar{E} \to \bar{Q}$ a skew-symmetric algebroid $(B^A_{\bar{E}}, \rho^A_{\bar{E}})$ given by

$$B^A_{\bar{E}}(\sigma, \bar{\sigma}) = \frac{1}{2}\left(B_{\bar{E}}(\sigma, \bar{\sigma}) - B_{\bar{E}}(\bar{\sigma}, \sigma)\right), \qquad \rho^A_{\bar{E}}(\sigma) = \frac{1}{2}\left(\rho^l_{\bar{E}}(\sigma) + \rho^r_{\bar{E}}(\sigma)\right),$$

and also a symmetric algebroid $(B^S_{\bar{E}}, \rho^S_{\bar{E}})$

$$B^S_{\bar{E}}(\sigma, \bar{\sigma}) = \frac{1}{2}\left(B_{\bar{E}}(\sigma, \bar{\sigma}) + B_{\bar{E}}(\bar{\sigma}, \sigma)\right) \qquad \rho^S_{\bar{E}}(\sigma) = \frac{1}{2}\left(\rho^l_{\bar{E}}(\sigma) - \rho^r_{\bar{E}}(\sigma)\right)$$

for $\sigma, \bar{\sigma} \in \Gamma(\tau_{\bar{E}})$.

Then, on a skew-symmetric tensor $T^A \in \Gamma(\tau_{\bigwedge^2 \bar{E}^*})$ the *skew-symmetric differential* $d^A_{\bar{E}}$ is defined by

$$\begin{aligned}(d^A_{\bar{E}} T^A)(\sigma, \bar{\sigma}, \bar{\bar{\sigma}}) \;=\;& \rho^A_{\bar{E}}(\sigma)(T^A(\bar{\sigma}, \bar{\bar{\sigma}})) - \rho^A_{\bar{E}}(\bar{\sigma})(T^A(\sigma, \bar{\bar{\sigma}})) + \rho^A_{\bar{E}}(\bar{\bar{\sigma}})(T^A(\sigma, \bar{\sigma}))\\ & - T^A(B^A_{\bar{E}}(\sigma, \bar{\sigma}), \bar{\bar{\sigma}}) + T^A(B^A_{\bar{E}}(\sigma, \bar{\bar{\sigma}}), \bar{\sigma}) - T^A(B^A_{\bar{E}}(\bar{\sigma}, \bar{\bar{\sigma}}), \sigma)\end{aligned}$$

and on a symmetric tensor T^S in $\Gamma(\tau_{\otimes^0_2 \bar{E}^*})$ the *symmetric differential* $d^S_{\bar{E}}$ is

$$\begin{aligned}(d^S_{\bar{E}} T^S)(\sigma, \bar{\sigma}, \bar{\bar{\sigma}}) \;=\;& \rho^S_{\bar{E}}(\sigma)(T^S(\bar{\sigma}, \bar{\bar{\sigma}})) + \rho^S_{\bar{E}}(\bar{\sigma})(T^S(\sigma, \bar{\bar{\sigma}})) + \rho^S_{\bar{E}}(\bar{\bar{\sigma}})(T^S(\sigma, \bar{\sigma}))\\ & - T^S(B^S_{\bar{E}}(\sigma, \bar{\sigma}), \bar{\bar{\sigma}}) - T^S(B^S_{\bar{E}}(\sigma, \bar{\bar{\sigma}}), \bar{\sigma}) - T^S(B^S_{\bar{E}}(\bar{\sigma}, \bar{\bar{\sigma}}), \sigma),\end{aligned}$$

for $\sigma, \bar{\sigma}, \bar{\bar{\sigma}} \in \Gamma(\tau_{\bar{E}})$.

Note that $d^A_{\bar{E}} T^A$ (respectively, $d^S_{\bar{E}} T^S$) is a skew-symmetric tensor of type (0,3) (respectively, a symmetric tensor of type (0,3)).

Now, we may extend the definition of the differential to any (0,2)-tensor in $\Gamma(\tau_{\otimes^0_2 \bar{E}^*})$. If T is a section of the vector bundle $\tau_{\otimes^0_2 \bar{E}^*} : \otimes^0_2 \bar{E}^* \to \bar{Q}$ then the differential of T is the section $d^{AS}_{\bar{E}} T$ of the vector bundle $\tau_{\otimes^0_3 \bar{E}^*} : \otimes^0_3 \bar{E}^* \to \bar{Q}$ defined by

$$(d^{AS}_{\bar{E}} T)(\sigma, \bar{\sigma}, \bar{\bar{\sigma}}) = (d^A_{\bar{E}} T^A)(\sigma, \bar{\sigma}, \bar{\bar{\sigma}}) + (d^S_{\bar{E}} T^S)(\sigma, \bar{\sigma}, \bar{\bar{\sigma}}) \tag{5.1}$$

where

$$T^A(\sigma, \bar{\sigma}) = \frac{1}{2}(T(\sigma, \bar{\sigma}) - T(\bar{\sigma}, \sigma)) \quad \text{and} \quad T^S(\sigma, \bar{\sigma}) = \frac{1}{2}(T(\sigma, \bar{\sigma}) + T(\bar{\sigma}, \sigma)),$$

for $\sigma, \bar{\sigma}, \bar{\bar{\sigma}} \in \Gamma(\tau_{\bar{E}})$. Note that T^A and T^S are the skew-symmetric and symmetric part of the tensor T.

Theorem 5.1. *Consider an algebroid structure $(B_E, \rho^l_E, \rho^r_E)$ over the vector bundle $\tau_E : E \to Q$ and the algebroid structure $(B_{\mathcal{T}^E_l E^*}, \rho^l_{\mathcal{T}^E_l E^*}, \rho^r_{\mathcal{T}^E_l E^*})$ induced over the vector bundle $\tau_{\mathcal{T}^E_l E^*} : \mathcal{T}^E_l E^* \to E^*$ as in (4.4) and (4.5). If the (1,3)-tensor R in equation (4.5) verifies the following relations*

$$R(\sigma, \bar{\sigma})\bar{\bar{\sigma}} = -R(\bar{\sigma}, \sigma)\bar{\bar{\sigma}} \tag{5.2}$$

and

$$R(\sigma,\bar\sigma)\bar{\bar\sigma}+R(\bar\sigma,\bar{\bar\sigma})\sigma+R(\bar{\bar\sigma},\sigma)\bar\sigma=0 \qquad \textit{(first Bianchi identity)}, \tag{5.3}$$

for all $\sigma,\bar\sigma,\bar{\bar\sigma}\in\Gamma(E)$, then the exact symplectic section $\Omega_{\mathcal{T}^E_l E^*}$ is closed, that is

$$d^A_{\mathcal{T}^E_l E^*}\Omega_{\mathcal{T}^E_l E^*}=0.$$

Proof. First, we are going to build the skew-symmetric algebroid induced on $\tau_{\mathcal{T}^E_l E^*}:\mathcal{T}^E_l E^*\to E^*$. The anchor map of this skew-symmetric algebroid is given by

$$\begin{aligned}\rho^A_{\mathcal{T}^E_l E^*}(\sigma_l^{(D^l)\mathbf{h}})&=\tfrac12(\sigma^{(D^l)\mathbf{h}}+\sigma^{(D^r)\mathbf{h}})\\ \rho^A_{\mathcal{T}^E_l E^*}(\kappa_l^{\mathbf{v}})&=\kappa^{\mathbf{v}}.\end{aligned}\tag{5.4}$$

and the skew-symmetric bracket is

$$\begin{aligned}&B^A_{\mathcal{T}^E_l E^*}(\sigma_l^{(D^l)\mathbf{h}},\bar\sigma_l^{(D^l)\mathbf{h}})=B^A_E(\sigma,\bar\sigma)_l^{(D^l)\mathbf{h}}+R(\sigma,\bar\sigma,\cdot)_l^{\mathbf{v}}\\ &B^A_{\mathcal{T}^E_l E^*}(\sigma_l^{(D^l)\mathbf{h}},\kappa_l^{\mathbf{v}})=\tfrac12(D^l_\sigma\kappa+D^r_\sigma\kappa)_l^{\mathbf{v}}=-B^A_{\mathcal{T}^E_l E^*}(\kappa_l^{\mathbf{v}},\sigma_l^{(D^l)\mathbf{h}})\\ &B^A_{\mathcal{T}^E_l E^*}(\kappa_l^{\mathbf{v}},\nu_l^{\mathbf{v}})=0\end{aligned}\tag{5.5}$$

for $\sigma,\bar\sigma\in\Gamma(\tau_E)$ and $\kappa,\nu\in\Gamma(\tau_{E^*})$.

To obtain the first equation of (5.5) we used the fact that the tensor R in (4.5) verifies the skew-symmetric property: $R(\sigma,\bar\sigma)=-R(\bar\sigma,\sigma)$.

In order to compute the skew-symmetric differential of the tensor $\Omega_{\mathcal{T}^E_l E^*}$, consider a local basis $\{\sigma_\alpha\}$ of $\Gamma(\tau_E)$ and the corresponding dual basis $\{\sigma^\alpha\}$ of $\Gamma(\tau_{E^*})$. A straightforward computation gives that

$$\begin{aligned}(d^A_{\mathcal{T}^E_l E^*}\Omega_{\mathcal{T}^E_l E^*})\left((\sigma_\alpha)_l^{(D^l)\mathbf{h}},(\sigma_\beta)_l^{(D^l)\mathbf{h}},(\sigma_\gamma)_l^{(D^l)\mathbf{h}}\right)&=R(\widehat{\sigma_\alpha,\sigma_\beta})\sigma_\gamma+R(\widehat{\sigma_\beta,\sigma_\gamma})\sigma_\alpha\\ &\quad+R(\widehat{\sigma_\gamma,\sigma_\alpha})\sigma_\beta=0, &(5.6)\\ (d^A_{\mathcal{T}^E_l E^*}\Omega_{\mathcal{T}^E_l E^*})\left((\sigma_\alpha)_l^{(D^l)\mathbf{h}},(\sigma_\beta)_l^{(D^l)\mathbf{h}},(\sigma^\gamma)_l^{\mathbf{v}}\right)&=0, &(5.7)\\ (d^A_{\mathcal{T}^E_l E^*}\Omega_{\mathcal{T}^E_l E^*})\left((\sigma_\alpha)_l^{(D^l)\mathbf{h}},(\sigma^\beta)_l^{\mathbf{v}},(\sigma^\gamma)_l^{\mathbf{v}}\right)&=0,\\ (d^A_{\mathcal{T}^E_l E^*}\Omega_{\mathcal{T}^E_l E^*})\left((\sigma^\alpha)_l^{\mathbf{v}},(\sigma^\beta)_l^{\mathbf{v}},(\sigma^\gamma)_l^{\mathbf{v}}\right)&=0,\end{aligned}$$

where for the proof of (5.7) we have used the fact that the bracket B^A_E can be written as

$$B^A_E(\sigma_\alpha,\sigma_\beta)=\frac12\left(D^l_{\sigma_\alpha}\sigma_\beta-D^l_{\sigma_\beta}\sigma_\alpha+D^r_{\sigma_\alpha}\sigma_\beta-D^r_{\sigma_\beta}\sigma_\alpha\right).$$

Since the (0,3)-tensor $d^A_{\mathcal{T}^E_l E^*}\Omega_{\mathcal{T}^E_l E^*}$ is skew-symmetric the proof is complete.

□

Natural choices of a $(1,3)$-tensor field R verifying properties (5.2) and (5.3) are $R\equiv 0$ and the curvature $R=R^{\nabla^{\mathcal{G}}}$ of a Levi-Civita connection in the case when E is a Lie algebroid with a bundle metric $\mathcal{G}$ (see (4.11) and (4.12)). From the last case, it is possible to construct new direct examples of a tensor field of type $(1,3)$ satisfying (5.2) and (5.3). Consider a Lie algebroid $\tau_E:E\to Q$ with Lie algebroid structure $([\![\cdot,\cdot]\!]_E,\rho_E)$ and $R_E=R^{\nabla^{\mathcal{G}}}$ the curvature of the Levi-Civita connection associated to a bundle metric $\mathcal{G}$. Take now a vector subbundle $\tau_D:D\to Q$ of E, $i_D:D\to E$ being

the canonical inclusion, equipped with an algebroid structure $(B_D, \rho_D^l, \rho_D^r)$ and an arbitrary vector bundle morphism $F : E \to D$. Then, we may construct the $(1,3)$-tensor field:

$$R_D(\sigma,\bar\sigma)\bar{\bar\sigma} = F(R^{\nabla^S}(i_D\circ\sigma, i_D\circ\bar\sigma)(i_D\circ\bar{\bar\sigma}))$$

for all $\sigma,\bar\sigma,\bar{\bar\sigma}\in\Gamma(D)$. It follows that R_D satisfies both conditions (5.2) and (5.3). Observe, for instance, that it is precisely the case of nonholonomic mechanics discussed in Subsection 2.2, where now F is the orthogonal projector P.

Remark 5.2. There is a natural extension of symmetric and skew-symmetric differentials on tensors of type $(0,k)$. That is, on $\Psi^A\in\Gamma(\tau_{\bigwedge^k\bar E^*})$ the *skew-symmetric differential* $d^A_{\bar E}$ of Ψ^A is a section of $\tau_{\bigwedge^{k+1}\bar E^*} : \bigwedge^{k+1}\bar E^*\to Q$ defined by

$$\begin{aligned}(d^A_{\bar E}\,\Psi^A)(\sigma_0,\sigma_1,...,\sigma_k) &= \sum_{i=1}^k(-1)^i\rho^A_{\bar E}(\sigma_i)(\Psi^A(\sigma_0,...,\widehat{\sigma}_i,...,\sigma_k))\\ &\quad+\sum_{i<j}(-1)^{i+j}\Psi^A(B^A_{\bar E}(\sigma_i,\sigma_j),\sigma_0,\sigma_1,...,\widehat{\sigma}_i,...,\widehat{\sigma}_j,...,\sigma_k)\end{aligned}$$

and on a symmetric tensor Ψ^S in $\Gamma(\tau_{\otimes^0_k\bar E^*})$ the *symmetric differential* $d^S_{\bar E}\Psi^S$ of Ψ^S is the symmetric tensor in $\Gamma(\tau_{\otimes^0_{k+1}\bar E^*})$ defined by

$$\begin{aligned}(d^S_{\bar E}\,\Psi^S)(\sigma_0,\sigma_1,...,\sigma_k) &= \sum_{i=1}^k\rho^S_{\bar E}(\sigma_i)(\Psi^S(\sigma_0,...,\widehat{\sigma}_i,...,\sigma_k))\\ &\quad-\sum_{i<j}\Psi^S(B^S_{\bar E}(\sigma_i,\sigma_j),\sigma_0,\sigma_1,...,\widehat{\sigma}_i,...,\widehat{\sigma}_j,...,\sigma_k),\end{aligned}$$

for $\sigma_0,\sigma_1,...,\sigma_k\in\Gamma(\tau_{\bar E})$.

Note that

$$d^A_{\bar E}(\Psi^A\wedge\mu^A) = d^A_{\bar E}\Psi^A\wedge\mu^A + (-1)^k\Psi^A\wedge d^A_{\bar E}\mu^A,\;\; d^S_{\bar E}(\Psi^S\vee\mu^S) = d^S_{\bar E}\Psi^S\vee\mu^S + \Psi^S\vee d^S_{\bar E}\mu^S,$$

for $\Psi^A\in\Gamma(\tau_{\Lambda^k\bar E^*})$, $\mu^A\in\Gamma(\tau_{\Lambda^l\bar E^*})$, $\Psi^S\in\Gamma(\tau_{\otimes^0_k\bar E^*})$, $\mu^S\in\Gamma(\tau_{\otimes^0_l\bar E^*})$, with Ψ^S and μ^S symmetric tensors and $\vee$ being the symmetric product. On the other hand, if $\alpha,\beta\in\Gamma(\tau_{\bar E^*})$ we have that

$$\alpha\otimes\beta = \frac{1}{2}(\alpha\wedge\beta+\alpha\vee\beta)$$

and thus

$$d^{AS}_{\bar E}(\alpha\otimes\beta) = \frac{1}{2}(d^A_{\bar E}\alpha\wedge\beta-\alpha\wedge d^A_{\bar E}\beta)+\frac{1}{2}(d^S_{\bar E}\alpha\vee\beta+\alpha\vee d^S_{\bar E}\beta).$$

Remark 5.3. (1) The *skew-symmetric differential* was defined in [23] as the *almost differential* on an *almost Lie algebroid*. Note that $(d^A_{\bar E})^2 = 0$ if and only if $(B^A_{\bar E},\rho^A_{\bar E})$ is a Lie algebroid structure on $\tau_{\bar E} : \bar E\to Q$.

(2) Let $\bar E$ be the tangent bundle of the manifold Q and ∇ be a linear connection on Q. Then, $(B^{\nabla}_{TQ}, id_{TQ}, -id_{TQ})$ is a symmetric algebroid structure on TQ, where

$$B^{\nabla}_{TQ}(X,Y) = \nabla_XY+\nabla_YX,\quad \text{for } X,Y\in\mathfrak{X}(Q).$$

Moreover, the corresponding symmetric differential d^S_{TQ} was considered in [20]. In fact, in [20] using the symmetric differential and the symmetric Lie derivative, the derivations of the algebra of symmetric tensors are classified and the Frölicher-Nijenhuis bracket for vector valued

symmetric tensors is introduced. This theory is the symmetric counterpart of the theory of vector valued differential forms which was developed by Frölicher-Nijenhuis [13].

6. Examples revisited

6.1. The symmetric case: Gradient extension of dynamical systems. (See Subsection 2.1).

In this case, we have a Riemannian manifold $(Q, \mathcal{G})$ and the vector bundle $\tau_{TQ}: TQ \to Q$ endowed with the symmetric product

$$B_{TQ}(X, Y) = \nabla^{\mathcal{G}}_X Y + \nabla^{\mathcal{G}}_Y X, \quad \text{for } X, Y \in \mathfrak{X}(Q).$$

The anchor maps are $\rho^l_{TQ} = id_{TQ}$ and $\rho^r_{TQ} = -id_{TQ}$. Thus,

$$B_{TQ}(X, Y) = D^l_X Y - D^r_X Y$$

where D^l (respectively, D^r) is the ρ^l_{TQ}-connection (respectively, the ρ^r_{TQ}-connection) defined by

$$D^l_X Y = \nabla^{\mathcal{G}}_X Y$$

(respectively, $D^r_X Y = -\nabla^{\mathcal{G}}_X Y$). Moreover, it is easy to prove that the TQ-tangent bundle to T^*Q, $\mathcal{T}^{TQ}_l T^*Q$, may be identified with the vector bundle $\tau_{T(T^*Q)}: T(T^*Q) \to T^*Q$. Under this identification, we have that (see (A.4))

$$\left(\frac{\partial}{\partial q^i}\right)_l^{D^l\mathbf{h}} = \frac{\partial}{\partial q^i} + \Gamma^k_{ij} p_k \frac{\partial}{\partial p_j}, \quad \left(\frac{\partial}{\partial q^i}\right)_l^{D^r\mathbf{h}} = -\frac{\partial}{\partial q^i} - \Gamma^k_{ij} p_k \frac{\partial}{\partial p_j}, \quad (dq^i)^{\mathbf{v}}_l = \frac{\partial}{\partial p_i}, \tag{6.1}$$

where (q^i, p_i) are fibred coordinates on T^*Q and Γ^k_{ij} are the Christoffel symbols of the Levi-Civita connection $\nabla^{\mathcal{G}}$. Therefore, using (4.8), we deduce that the exact symplectic structure $\Omega_{T(T^*Q)}$ is just the canonical symplectic structure of T^*Q

$$\Omega_{T(T^*Q)} = dq^i \wedge dp_i \tag{6.2}$$

(note that $\Gamma^k_{ij} = \Gamma^k_{ji}$).

On the other hand, from (4.4) and (6.1), it follows that

$$\rho^r_{T(T^*Q)}\left(\frac{\partial}{\partial q^i}\right) = -\frac{\partial}{\partial q^i} - 2\Gamma^k_{ij} p_k \frac{\partial}{\partial p_j}, \quad \rho^r_{T(T^*Q)}\left(\frac{\partial}{\partial p_i}\right) = \frac{\partial}{\partial p_i}.$$

Consequently, if $H \in C^\infty(T^*Q)$ we obtain that

$$d^r_{T(T^*Q)} H = \left(-\frac{\partial H}{\partial q^i} - 2\Gamma^k_{ij} p_k \frac{\partial H}{\partial p_j}\right) dq^i + \frac{\partial H}{\partial p_j} dp_j$$

which implies that the right-Hamiltonian section of H is the vector field on T^*Q given by

$$\mathcal{H}_H^{(\Omega_{T(T^*Q)}, r)} = \frac{\partial H}{\partial p_i} \frac{\partial}{\partial q^i} + \left(\frac{\partial H}{\partial q^i} + 2\Gamma^k_{ij} p_k \frac{\partial H}{\partial p_i}\right) \frac{\partial}{\partial p_i}.$$

Thus, if we apply the above construction to the Hamiltonian function $H = H_X = \hat{X}$, with $X \in \mathfrak{X}(Q)$, we reobtain the Hamilton equations (2.6).

6.2. **Skew-symmetric mechanics: Nonholonomic systems.** (See Subsection 2.2).

Consider $(\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}})$ a Lie algebra of finite dimension. In this case, the Lie bracket is the Lie algebra structure $[\cdot,\cdot]_{\mathfrak{g}}$ and the anchor map is the null map.

Consider now a nonholonomic mechanical system on $\mathfrak{g}$, that is a vector subspace $\mathfrak{d} \subset \mathfrak{g}$ of kinematic constraints ($\mathfrak{d}$ is not, in general, a Lie subalgebra) and a lagrangian function $L : \mathfrak{g} \to \mathbb{R}$ of mechanical type induced by a scalar product $\mathcal{G}$ on $\mathfrak{g}$. As we did in Example 2.2 we assert that $(\mathfrak{d}, [\cdot,\cdot]_{\mathfrak{d}}, 0)$ is a skew-symmetric algebroid with the bracket given by $[\xi,\eta]_{\mathfrak{d}} = P([i_{\mathfrak{d}}(\xi), i_{\mathfrak{d}}(\eta)]_{\mathfrak{g}})$ (see Subsection 2.2).

In what follows we are going to use the formalism in $\mathcal{T}^{\mathfrak{d}}\mathfrak{d}^*$ proposed in Section 4 to find an exact symplectic form and the corresponding Hamilton equations.

Let us consider a basis $\{\xi_a\}$ of $\mathfrak{d}$ and $\{\xi^a\}$ the dual basis of $\mathfrak{d}^*$.

In this case, we choose the 0-connection $D = D^l = D^r$ to be $D_{\xi_a}\xi_b = \frac{1}{2}[\xi_a, \xi_b]_{\mathfrak{d}}$ and thus $\Gamma^c_{ab} = \frac{1}{2}c^c_{ab}$ where c^c_{ab} are the structure constants of the skew-symmetric algebroid $(\mathfrak{d}, [\cdot,\cdot]_{\mathfrak{d}}, 0)$.

Now, it is easy to prove that $\mathcal{T}^{\mathfrak{d}}\mathfrak{d}^*$ may be identified with $\mathfrak{d}^* \times \mathfrak{d} \times \mathfrak{d}^*$ and, under this identification, the vector bundle projection

$$\tau_{\mathcal{T}^{\mathfrak{d}}\mathfrak{d}^*} : \mathcal{T}^{\mathfrak{d}}\mathfrak{d}^* \to \mathfrak{d}^*$$

is just the canonical projection on the first factor

$$pr_1 : \mathfrak{d}^* \times \mathfrak{d} \times \mathfrak{d}^* \to \mathfrak{d}^*.$$

Since $\mathfrak{d}$ satisfies the hypotheses of Remark 4.5, it is easy to see that a suitable structure of skew-symmetric algebroid on $\mathcal{T}^{\mathfrak{d}}\mathfrak{d}^* \simeq \mathfrak{d}^* \times \mathfrak{d} \times \mathfrak{d}^* \to \mathfrak{d}^*$ is determined by the following relations:

$$B_{\mathcal{T}^{\mathfrak{d}}\mathfrak{d}^*}\left((\cdot,\sigma,\upsilon),(\cdot,\sigma',\upsilon')\right)(\kappa) = (\kappa, [\sigma,\sigma']_{\mathfrak{d}}, 0)$$

for $\kappa \in \mathfrak{d}^*$, $(\sigma,\upsilon),(\sigma',\upsilon') \in \mathfrak{d} \times \mathfrak{d}^*$ and

$$\rho^l_{\mathcal{T}^{\mathfrak{d}}\mathfrak{d}^*}(\kappa,\sigma,\upsilon) = (\kappa,\upsilon).$$

A straightforward computation shows that

$$\Omega_{\mathcal{T}^{\mathfrak{d}}\mathfrak{d}^*}\left((\kappa,\sigma,\upsilon),(\kappa,\sigma',\upsilon')\right) = \upsilon'(\sigma) - \upsilon(\sigma') - \kappa\left([\sigma,\sigma']_{\mathfrak{d}}\right)$$

for $(\kappa,\sigma,\upsilon),(\kappa,\sigma',\upsilon') \in \mathfrak{d}^* \times \mathfrak{d} \times \mathfrak{d}^*$.

Now, we consider the basis $\{E_a, \tilde{E}^a\}$ of $\Gamma(\tau_{\mathcal{T}^{\mathfrak{d}}\mathfrak{d}^*})$ defined as

$$\begin{array}{rcl} E_a &=& (\cdot,\xi_a,0) \quad \text{such that} \quad E_a(\kappa) = (\kappa,\xi_a,0) \\ \tilde{E}^a &=& (\cdot,0,\xi^a) \quad \text{such that} \quad \tilde{E}^a(\kappa) = (\kappa,0,\xi^a). \end{array}$$

Then, we obtain

$$B_{\mathcal{T}^{\mathfrak{d}}\mathfrak{d}^*}(E_a,E_b) = c^c_{ab}E_c$$

$$B_{\mathcal{T}^{\mathfrak{d}}\mathfrak{d}^*}(E_a,\tilde{E}^b) = B_{\mathcal{T}^{\mathfrak{d}}\mathfrak{d}^*}(\tilde{E}^a,E_b) = B_{\mathcal{T}^{\mathfrak{d}}\mathfrak{d}^*}(\tilde{E}^a,\tilde{E}^b) = 0$$

and the anchor map is

$$\rho_{\mathcal{T}^{\mathfrak{d}}\mathfrak{d}^*}(E_a) = 0 \;\text{ and }\; \rho_{\mathcal{T}^{\mathfrak{d}}\mathfrak{d}^*}(\tilde{E}^b) = \xi^b.$$

Thus we conclude that

$$\Omega_{\mathcal{T}^{\mathfrak{d}}\mathfrak{d}^*} = -\frac{1}{2}c^c_{ab}p_cE^a \wedge E^b + E^a \wedge \tilde{E}_a$$

where $\{E^a, \tilde{E}_b\}$ is the dual basis induced by $\{E_a, \tilde{E}^b\}$ and p_a are the coordinates in $\mathfrak{d}^*$ induced by ξ^a. Moreover,

$$d_{\mathcal{T}^{\mathfrak{d}}\mathfrak{d}^*}H(E_a) = 0 \quad \text{and} \quad d_{\mathcal{T}^{\mathfrak{d}}\mathfrak{d}^*}H(\tilde{E}^a) = \frac{\partial H}{\partial p_a}.$$

Therefore, we have that the unique solution of Equation

$$i_X \Omega_{\mathcal{T}^{\mathfrak{d}}\mathfrak{d}^*} = d_{\mathcal{T}^{\mathfrak{d}}\mathfrak{d}^*}H\,, \tag{6.3}$$

is

$$\mathcal{H}_H^{\Omega_{\mathcal{T}^{\mathfrak{d}}\mathfrak{d}^*}} = \frac{\partial H}{\partial p_a}E_a + c_{ab}^c p_c \frac{\partial H}{\partial p_b}\tilde{E}^a$$

Now, using the anchor map we obtain that the corresponding Hamiltonian vector field on $\mathfrak{d}^*$:

$$\mathcal{H}_H^{\Pi_{\mathfrak{d}^*}} = c_{ab}^c p_c \frac{\partial H}{\partial p_b}\xi^a = \rho_{\mathcal{T}^{\mathfrak{d}}\mathfrak{d}^*}(\mathcal{H}_H^{\Omega_{\mathcal{T}^{\mathfrak{d}}\mathfrak{d}^*}}).$$

Its integral curves are precisely the *nonholonomic Lie-Poisson equations* (see [9] and references therein)

$$\dot{p}_a = c_{ab}^c p_c \frac{\partial H}{\partial p_b},$$

that is, using a classical notation,

$$\dot{\kappa} = ad^{\mathfrak{d}^*}_{\frac{\partial H}{\partial \kappa}}\kappa, \quad \text{for } \kappa \in \mathfrak{d}^*$$

where $ad^{\mathfrak{d}^*} : \mathfrak{d} \times \mathfrak{d}^* \to \mathfrak{d}^*$ is the map defined as $(ad^{\mathfrak{d}^*}_\xi(\kappa))(\eta) = \kappa([\xi, \eta]_{\mathfrak{d}})$ for $\xi, \eta \in \mathfrak{d}$ and $\kappa \in \mathfrak{d}^*$. Note that if $\mathfrak{d} = \mathfrak{g}$ then $ad^{\mathfrak{d}^*} = ad^*$ is the infinitesimal coadjoint representation.

6.3. **Mixed mechanics:**

6.3.1 **Generalized nonholonomic mechanics on a Lie algebra**. (See Subsection 2.3.1).

As in the previous example, consider a Lie algebra $(\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}})$ of finite dimension, a subspace $\mathfrak{d} \subset \mathfrak{g}$ and a lagrangian $L : \mathfrak{g} \to \mathbb{R}$ of mechanical type induced by a scalar product $\mathcal{G}$ on $\mathfrak{g}$. Since we are considering a generalized nonholonomic system, $\mathfrak{d}$ is endowed with an algebroid structure $(\mathfrak{d}, B_{\mathfrak{d}}, 0, 0)$ given by (2.18), (2.19) and (2.20), (in this case, the anchors are zero but the bracket is not necessarily skew-symmetric). In fact,

$$B_{\mathfrak{d}}(\sigma, \sigma') = P[\sigma, \Pi(\sigma')]_{\mathfrak{g}}$$

for $\sigma, \sigma' \in \mathfrak{d}$ and $P : \mathfrak{g} = \mathfrak{d} \oplus \mathfrak{d}^\perp \to \mathfrak{d} \subseteq \mathfrak{g}$, $\Pi : \mathfrak{g} = \tilde{\mathfrak{d}} \oplus \mathfrak{d}^\perp \to \tilde{\mathfrak{d}} \subseteq \mathfrak{g}$ the corresponding projectors.

As in the previous example, the space $\mathcal{T}^{\mathfrak{d}}\mathfrak{d}^*$ may be identifiaed with the product $\mathfrak{d}^* \times \mathfrak{d} \times \mathfrak{d}^*$ and, under this identification, the vector bundle projection is the canonical projection on the first factor

$$pr_1 : \mathfrak{d}^* \times \mathfrak{d} \times \mathfrak{d}^* \to \mathfrak{d}^*.$$

Then, we obtain that a suitable bracket on $\Gamma(\mathcal{T}^{\mathfrak{d}}\mathfrak{d}^*)$ has the following form

$$B_{\mathcal{T}^{\mathfrak{d}}\mathfrak{d}^*}\left((\cdot, \sigma, \upsilon), (\cdot, \sigma', \upsilon')\right)(\kappa) = (\kappa, B_{\mathfrak{d}}(\sigma, \sigma'), R(\kappa, \sigma, \upsilon, \sigma', \upsilon')),$$

for $\kappa \in \mathfrak{d}^*$, $(\sigma, \upsilon), (\sigma', \upsilon') \in \mathfrak{d}^*$, with $R(\kappa, \sigma, \upsilon, \sigma', \upsilon') \in \mathfrak{d}^*$. The anchor maps, in this case, are

$$\begin{aligned} \rho^l_{\mathcal{T}^{\mathfrak{d}}\mathfrak{d}^*}(\kappa, \sigma, \upsilon) &= (\kappa, \upsilon) \\ \rho^r_{\mathcal{T}^{\mathfrak{d}}\mathfrak{d}^*}(\kappa, \sigma, \upsilon) &= \Big(\kappa, \upsilon + i^*_{\mathfrak{d}}(\Pi^*(\nabla^{\mathcal{G}}_\sigma P^*\kappa) - \nabla^{\mathcal{G}}_{\Pi(\sigma)}P^*\kappa)\Big) \end{aligned}$$

where $i_{\mathfrak{d}} : \mathfrak{d} \longrightarrow \mathfrak{g}$ is the canonical inclusion and $\nabla^{\mathcal{G}}$ is the Levi-Civita connection of the scalar product $\mathcal{G}$ on $\mathfrak{g}$. Thus,

$$\Omega_{\mathcal{J}^{\mathfrak{d}}\mathfrak{d}^*}((\kappa, \sigma, \upsilon), (\kappa, \sigma', \upsilon')) = \upsilon'(\sigma) - \upsilon(\sigma') - \kappa\left(P(\nabla^{\mathcal{G}}_{\sigma'}\Pi(\sigma) - \nabla^{\mathcal{G}}_{\sigma}\Pi(\sigma'))\right)$$

for $(\kappa, \sigma, \upsilon), (\kappa, \sigma', \upsilon') \in \mathfrak{d}^* \times \mathfrak{d} \times \mathfrak{d}^*$.

Considering the same basis as in the previous example $\{E_a, \tilde{E}^b\}$ (but with $\{\xi_a, \xi_A\}$ an adapted basis to $\mathfrak{d} \oplus \tilde{\mathfrak{d}}^{\perp}$) and the dual basis $\{E^a, \tilde{E}_b\}$ we get that the left and right anchor maps are, respectively

$$\rho^l_{\mathcal{J}^{\mathfrak{d}}\mathfrak{d}^*}(E_a) = 0 \quad \text{and} \quad \rho^l_{\mathcal{J}^{\mathfrak{d}}\mathfrak{d}^*}(\tilde{E}^b) = \xi^b,$$

$$\rho^r_{\mathcal{J}^{\mathfrak{d}}\mathfrak{d}^*}(E_a) = -\Xi^c_{ab}p_c\xi^b \quad \text{and} \quad \rho^r_{\mathcal{J}^{\mathfrak{d}}\mathfrak{d}^*}(\tilde{E}^b) = \xi^b,$$

where $\Xi^c_{ab} = (D^l)^c_{ab} - (D^r)^c_{ab}$, with $(D^l)^c_{ab}$ and $(D^r)^c_{ab}$ the Christoffel symbols of the left and right connections for the generalized nonholonomic systems (see Appendix B).

A similar computation, as in the nonholonomic case, shows that

$$\Omega_{\mathcal{J}^{\mathfrak{d}}\mathfrak{d}^*} = \frac{1}{2}(-\tilde{c}^c_{ab}p_c + \Xi^c_{ba})E^a \wedge E^b + E^a \wedge \tilde{E}_a.$$

Finally by means of the left-right Hamiltonian vector field of H, we get

$$\dot{p}_a = \tilde{c}^c_{ab}p_c\frac{\partial H}{\partial p_b}.$$

Therefore, we obtain *the generalized nonholonomic Lie-Poisson equations*,

$$\dot{\kappa} = ad^{\mathfrak{d}^*}_{\frac{\partial H}{\partial \kappa}}\kappa$$

for $\kappa \in \mathfrak{d}^*$ and where $ad^{\mathfrak{d}^*} : \mathfrak{d} \times \mathfrak{d}^* \longrightarrow \mathfrak{d}^*$ is the map defined as $(ad^{\mathfrak{d}^*}_{\xi}(\kappa))(\eta) = \kappa(B_{\mathfrak{d}}(\xi, \eta))$, for $\xi, \eta \in \mathfrak{d}$ and $\kappa \in \mathfrak{d}^*$.

6.3.2 **Lagrangian mechanics for modifications of the standard Lie bracket.**

Let us reconsider the Example in Subsection 2.3.2, in the case when the tensor field T is given by $T(X,Y) = S(X,Y) - S(Y,X)$, for $X, Y \in \mathfrak{X}(Q)$, where S is the contorsion tensor field induced by an affine connection ∇. The horizontal and vertical lifts induced by the connection ∇ give rise to an almost Lie algebroid structure on $TT^*Q \to T^*Q$. Straightforward computations permit to deduce that

$$\Omega_{TT^*Q} = dq^i \wedge (dp_i - S^k_{ji}p_k dq^j).$$

Since

$$d^r_{TT^*Q}H = \frac{\partial H}{\partial q^i}dq^i + \frac{\partial H}{\partial p_i}dp_i,$$

then the hamiltonian vector field

$$i_{\mathcal{H}_H^{(\Omega_{T(T^*Q)},lr)}}\Omega_{TT^*Q} = d^r_{TT^*Q}H \tag{6.4}$$

is

$$\mathcal{H}_H^{(\Omega_{T(T^*Q)},lr)} = \frac{\partial H}{\partial p_i}\frac{\partial}{\partial q^i} - \left(\frac{\partial H}{\partial q^i} - (S^k_{ji} - S^k_{ij})p_k\frac{\partial H}{\partial p_k}\right)\frac{\partial}{\partial p_i}.$$

Thus, the integral curves of $\mathcal{H}_H^{(\Omega_{T(T^*Q)},lr)}$ are just the solutions of Eqs.(2.23). Observe that Equation (6.4) exactly reproduces the almost symplectic realization of generalized Chaplygin systems (see [8]).

7. Conclusions and future work

A symplectic realization of the Hamiltonian dynamics on an algebroid is derived. In fact, we prove that Hamiltonian systems on an algebroid can be described by a symplectic equation constructed in the same way than in the standard one. For this purpose, the theory of generalized connections on an anchored vector bundle $\tau_E : E \to Q$ is widely used. In particular, we used the corresponding theory of horizontal and vertical lifts of tensor fields on $\tau_E : E \to Q$ to vector fields on the dual vector bundle E^*. Taking into account that there exists a lot of examples of Hamiltonian systems on an algebroid (gradient systems, nonholonomic mechanical systems, generalized nonholonomic mechanical systems,...), the above results show the ubiquity of the symplectic Hamiltonian equations in Mechanics.

In this paper, we suppose that the constraints (kinematic or variational) are linear. It would be interesting to discuss the more general case when the constraints are not linear and, more precisely, the case of affine constraints.

Another goal we have proposed is to develop a Klein formalism for Lagrangian systems on algebroids.

Finally, a different aspect on which we intend to work is a Hamilton-Jacobi theory for Hamiltonian systems on algebroids.

Appendix A: anchored vector bundles, connections and horizontal (vertical) lifts

Definition A.1. [5] *An anchored vector bundle is a real vector bundle $\tau_E : E \to Q$ over a manifold Q and a vector bundle morphism $\rho_E : E \to TQ$. The map $\rho_E : E \to TQ$ is called the anchor map of the anchored vector bundle.*

Now, suppose that $(\tau_E : E \to Q, \rho_E)$ is an anchored vector bundle over Q and denote by $\Gamma(\tau_E)$ the space of C^∞-sections of the vector bundle $\tau_E : E \to Q$.

Definition A.2. [5] *A ρ_E-connection on the anchored vector bundle $(\tau_E : E \to Q, \rho_E)$ is a $\mathbb{R}$-bilinear map $D : \Gamma(\tau_E) \times \Gamma(\tau_E) \to \Gamma(\tau_E)$ such that*

$$D_{f\sigma}\gamma = fD_\sigma\gamma \quad \text{and} \quad D_\sigma(g\gamma) = \rho_E(\sigma)(g)\gamma + gD_\sigma\gamma \tag{A.1}$$

for $f \in C^\infty(Q)$, and $\sigma, \gamma \in \Gamma(\tau_E)$.

Remark A.3. Every vector bundle $\tau_E : E \to Q$ admits a ρ_E-connection. In fact, let $\nabla : \mathfrak{X}(Q) \times \Gamma(\tau_E) \to \Gamma(\tau_E)$ be an standard linear connection on $\tau_E : E \to Q$. Then, if we define the map $D : \Gamma(\tau_E) \times \Gamma(\tau_E) \to \Gamma(\tau_E)$ as

$$D_\sigma\gamma = \nabla_{\rho_E(\sigma)}\gamma \quad \text{for } \sigma, \gamma \in \Gamma(\tau_E),$$

it is easy to prove that D is a ρ_E-connection.

Let D be a ρ_E-connection on the anchored vector bundle $(\tau_E : E \to Q, \rho_E)$. If (q^i) are local coordinates on Q and $\{\sigma_\alpha\}$ is a local basis of $\Gamma(\tau_E)$ then

$$D_{f^\alpha\sigma_\alpha}(g^\beta\sigma_\beta) = \left(f^\alpha g^\beta D^\gamma_{\alpha\beta} + f^\alpha (\rho_E)^i_\alpha \frac{\partial g^\gamma}{\partial q^i} \right) \sigma_\gamma$$

for $f^\alpha, g^\beta \in C^\infty(Q)$, where

$$\rho_E(\sigma_\alpha) = (\rho_E)^i_\alpha \frac{\partial}{\partial q^i} \quad \text{and} \quad D_{\sigma_\alpha}\sigma_\beta = D^\gamma_{\alpha\beta}\sigma_\gamma.$$

$D^\gamma_{\alpha\beta}$ are the Christoffel symbols of the connection D with respect to the local basis $\{\sigma_\alpha\}$.

Now, suppose that σ_q is an element of the fiber E_q, with $q \in Q$. Then, we may introduce the $\mathbb{R}$-linear map $D_{\sigma_q} : \Gamma(\tau_E) \to E_q$ given by

$$D_{\sigma_q}\gamma = (D_\sigma\gamma)(q), \quad \text{for } \gamma \in \Gamma(\tau_E),$$

where $\sigma \in \Gamma(\tau_E)$ and $\sigma(q) = \sigma_q$. Note that, using (A.1), one deduces that the map D_{σ_q} is well defined. Thus, if $\kappa_q \in E^*_q$, we may consider the linear map

$$\,^{D\mathbf{h}}_{\kappa_q} : E_q \to T_{\kappa_q}E^*, \quad \sigma_q \mapsto (\sigma_q)^{D\mathbf{h}}_{\kappa_q}$$

where $(\sigma_q)^{D\mathbf{h}}_{\kappa_q}$ is the tangent vector to E^* at κ_q which is characterized by the following conditions

$$(\sigma_q)^{D\mathbf{h}}_{\kappa_q}(f \circ \tau_{E^*}) = \rho_E(\sigma_q)(f) \;\text{ and }\; (\sigma_q)^{D\mathbf{h}}_{\kappa_q}(\widehat{\gamma}) = \kappa_q(D_{\sigma_q}\gamma), \tag{A.2}$$

for $f \in C^\infty(Q)$ and $\gamma \in \Gamma(\tau_E)$. Here, $\widehat{\gamma} : E^* \to \mathbb{R}$ is the linear function on E^* induced by the section γ.

In particular, if $\sigma \in \Gamma(\tau_E)$ we may define the *D-horizontal lift to E^** as the vector field $\sigma^{D\mathbf{h}}$ on E^* given by

$$\sigma^{D\mathbf{h}}(\kappa_q) = (\sigma(q))^{D\mathbf{h}}_{\kappa_q}, \quad \text{for } \kappa_q \in E^*_q, \text{ with } q \in Q.$$

It is clear that

$$(\sigma + \sigma')^{D\mathbf{h}} = \sigma^{D\mathbf{h}} + (\sigma')^{D\mathbf{h}}, \; (f\sigma)^{D\mathbf{h}} = (f \circ \tau_{E^*})\sigma^{D\mathbf{h}}, \tag{A.3}$$

for $\sigma, \sigma' \in \Gamma(\tau_E)$ and $f \in C^\infty(Q)$.

Moreover, if (q^i) are local coordinates on Q and $\{\sigma_\alpha\}$ is a local basis of $\Gamma(\tau_E)$, then we have the corresponding local coordinates (q^i, p_α) on E^* and

$$\sigma^{D\mathbf{h}}_\alpha = (\rho_E)^i_\alpha \frac{\partial}{\partial q^i} + D^\gamma_{\alpha\beta}p_\gamma \frac{\partial}{\partial p_\beta}, \tag{A.4}$$

(for more details, see [5]).

On the other hand, if $\kappa'_q \in E^*_q$ we may consider the *standard vertical lift* as the linear map

$$\,^{\mathbf{v}}_{\kappa'_q} : E^*_q \to T_{\kappa'_q}E^*, \quad \kappa_q \mapsto (\kappa_q)^{\mathbf{v}}_{\kappa'_q}$$

with $(\kappa_q)^{\mathbf{v}}_{\kappa'_q}$ being the tangent vector to E^* at κ'_q which is characterized by the following conditions

$$(\kappa_q)^{\mathbf{v}}_{\kappa'_q}(f \circ \tau_{E^*}) = 0 \quad \text{and} \quad (\kappa_q)^{\mathbf{v}}_{\kappa'_q}(\widehat{\gamma}) = \kappa_q(\gamma(q)) \tag{A.5}$$

for $f \in C^\infty(Q)$ and $\gamma \in \Gamma(\tau_E)$.

Thus, if $\kappa \in \Gamma(\tau_{E^*})$ is a section of the dual vector bundle $\tau_{E^*} : E^* \to Q$ then the *vertical lift to* E^* is the vector field $\kappa^{\mathbf{v}}$ on E^* given by

$$\kappa^{\mathbf{v}}(\kappa'_q) = (\kappa(q))^{\mathbf{v}}_{\kappa'_q} \quad \text{for } \kappa'_q \in E^*_q, \text{ with } q \in Q.$$

It is clear that

$$(\kappa + \kappa')^{\mathbf{v}} = \kappa^{\mathbf{v}} + (\kappa')^{\mathbf{v}}, \; (f\kappa)^{\mathbf{v}} = (f \circ \tau_{E^*})\kappa^{\mathbf{v}}, \tag{A.6}$$

for $\kappa, \kappa' \in \Gamma(\tau_{E^*})$ and $f \in C^\infty(Q)$.

Moreover, if (q^i) are local coordinates on Q, $\{\sigma_\alpha\}$ is a local basis of $\Gamma(\tau_E)$, $\{\sigma^\alpha\}$ is the dual basis of $\Gamma(\tau_{E^*})$ and (q^i, p_α) the corresponding local coordinates on E^* then

$$(\sigma^\alpha)^{\mathbf{v}} = \frac{\partial}{\partial p_\alpha}. \tag{A.7}$$

Remark A.4. If $\{\sigma_\alpha\}$ is a local basis of $\Gamma(\tau_E)$ and $\{\sigma^\alpha\}$ is the dual basis of $\Gamma(\tau_{E^*})$ then $\{\sigma_\alpha^{D\mathbf{h}}, (\sigma^\alpha)^{\mathbf{v}}\}$ is not, in general, a local basis of vector fields on E^*. Note that ρ_E is not, in general, an epimorphism.

Remark A.5. The ρ_E-connection D induces a ρ_E-connection D^* on the dual vector bundle τ_{E^*} : $E^* \to Q$ which is defined by

$$(D^*_\sigma \kappa)(\gamma) = \rho_E(\sigma)(\kappa(\gamma)) - \kappa(D_\sigma \gamma),$$

for $\sigma, \gamma \in \Gamma(\tau_E)$ and $\kappa \in \Gamma(\tau_{E^*})$. If $\{\sigma_\alpha\}$ is a local basis of $\Gamma(\tau_E)$ and $\{\sigma^\alpha\}$ is the dual basis of $\Gamma(\tau_{E^*})$ then $D^*_{\sigma_\alpha}\sigma^\gamma = -D^\gamma_{\alpha\beta}\sigma^\beta$, where $D^\gamma_{\alpha\beta}$ are the Christoffel symbols of the connection D. Therefore, if $\sigma \in \Gamma(\tau_E)$ it is possible to consider the corresponding D^*- horizontal lift to E as a vector field $\sigma^{D^*\mathbf{h}}$ on E.

The above results are a generalization of some lifting operations previously defined in [35, 36, 37] for the case $E = TQ$ and $\rho_E = \rho_{TQ} = id_{TQ}$.

Appendix B: generalized nonholonomic systems

Let us consider a vector bundle $\tau_E : E \to Q$ with a Lie algebroid structure $([\![\cdot, \cdot]\!]_E, \rho_E)$. A *linear generalized nonholonomic system* on E is a mechanical system determined by a regular lagrangian function $L : E \to \mathbb{R}$ and two distributions, the kinematic constraints described by a vector subbundle $\tau_D : D \to Q$ and the variational constraints given by the vector subbundle $\tau_{\tilde{D}} : \tilde{D} \to Q$. As we have explained in section 2.3.1, the distribution $\tilde{D}$ is the subspace where the constraint forces are doing null work. It is clear that in the (classical) nonholonomic systems $D = \tilde{D}$. Generalized nonholonomic systems were studied in [2, 6, 7, 29].

We will assume that the lagrangian is of mechanical type, that is, we have a bundle metric $\mathcal{G}$ on E and a real C^∞-function $V : Q \to \mathbb{R}$ such that

$$L(e) = \frac{1}{2}\mathcal{G}(e, e) - V(\tau_E(e)), \quad \text{for } e \in E.$$

Moreover, we will assume that the following compatibility condition holds

$$E = D \oplus \tilde{D}^\perp$$

where $\tilde{D}^{\perp}$ is the orthogonal complement of the variational distribution $\tilde{D}$ with respect to the bundle metric $\mathcal{G}$.

We have that the equations of motion of such a system are given by $\delta L_{\gamma(t)} \in \tilde{D}^0_{\tau_D(\gamma(t))}$, for a ρ_E-admissible curve $\gamma : I \to D$. Then, the equations of motion are

$$\begin{cases} \dfrac{dq}{dt} = \rho_E \circ \gamma, \\ \nabla^{\mathcal{G}}_{\gamma(t)}\gamma(t) + grad_{\mathcal{G}} V(q(t)) \in \tilde{D}^{\perp}_{q(t)}, \\ \gamma(t) \in D_{q(t)}, \end{cases} \tag{B.1}$$

where $\nabla^{\mathcal{G}}$ is the Levi-Civita connection of $\mathcal{G}$, $\mathrm{grad}_{\mathcal{G}}(V)$ is the section of $\tau_E : E \to Q$ given by

$$\mathcal{G}(\mathrm{grad}_{\mathcal{G}}(V), \sigma) = \rho_E(\sigma)(V), \quad \text{for } \sigma \in \Gamma(\tau_E),$$

and $q = \tau_D \circ \gamma$.

Suppose that (q^i) are local coordinates on an open subset U of Q and that $\{\sigma_\alpha\} = \{\sigma_a, \sigma_A\}$ is a basis of sections of the vector bundle $\tau_E^{-1}(U) \to U$ adapted to the decomposition $E = D \oplus \tilde{D}^{\perp}$. We will denote by $(q^i, v^\alpha) = (q^i, v^a, v^A)$ the corresponding local coordinates on E. We will assume that the bundle metric $\mathcal{G}$ can be locally written as $\mathcal{G} = \mathcal{G}_{\alpha\beta}\sigma^\alpha \otimes \sigma^\beta$. We will also assume that σ_a (respectively, σ_A) is an orthonormal basis of $\Gamma(\tau_D)$ (respectively, $\Gamma(\tau_{\tilde{D}^{\perp}})$). Thus, we have that $\mathcal{G}_{ab} = \delta_a^b$ (respectively $\mathcal{G}_{AB} = \delta_A^B$) and it is easy to see that $\tilde{D} = span\{\sigma_d - \mathcal{G}_{dA}\sigma_A\}$. Then the system (B.1) can be written for $\gamma = v^a\sigma_a + v^A\sigma_A$ and $q = \tau_E \circ \gamma$

$$\begin{cases} \mathcal{G}(\nabla^{\mathcal{G}}_{\gamma(t)}\gamma(t) + grad_{\mathcal{G}} V(q(t)), \sigma_d - \mathcal{G}_{dB}\sigma_B) = 0 \\ \\ \dot{q}^i = (\rho^l_D)^i_a v^a, \;\; v^A = 0. \end{cases} \tag{B.2}$$

A straightforward computation shows that the system (B.2) is equivalent to

$$\begin{cases} \dot{v}^c + v^a v^b \Gamma^c_{ab} + (\rho^r_D)^i_c \frac{\partial V}{\partial q^i} = 0 \\ \dot{q}^i = (\rho^l_D)^i_a v^a \end{cases} \tag{B.3}$$

where Γ^c_{ab} are the Christoffel symbols of the Levi-Civita connection in $\tau_E : E \to Q$ and

$$\begin{aligned} (\rho^l_D)^i_a &= (\rho_E)^i_a \\ (\rho^r_D)^i_c &= \mathcal{G}^{cd}((\rho_E)^i_d - \mathcal{G}_{dA}(\rho_E)^i_A) \end{aligned} \tag{B.4}$$

with $\mathcal{G}^{\alpha\beta}$ the inverse matrix of $\mathcal{G}_{\alpha\beta}$ (note that $\mathcal{G}^{eC} = -\mathcal{G}^{ef}\mathcal{G}_{fC}$ and that $\mathcal{G}^{ef} + \mathcal{G}^{eC}\mathcal{G}_{Cf} = \delta^e_f$).

Now we can write these symbols Γ^c_{ab} in terms of the local structure functions of the Lie algebroid $\tau_E : E \to Q$ using the expression

$$\Gamma^c_{ab} = \frac{1}{2}\mathcal{G}^{c\alpha}\left([\alpha, a; b] + [\alpha, b; a] + [a, b; \alpha]\right)$$

where $[\alpha, \beta; \gamma] = \frac{\partial \mathcal{G}_{\alpha\beta}}{\partial q^i}(\rho_E)^i_\gamma + C^\mu_{\alpha\beta}\mathcal{G}_{\mu\gamma}$ (see [9, 10]). Then, since $\mathcal{G}^{cA} = -\mathcal{G}^{cd}\mathcal{G}_{dA}$, it is easy to prove that

$$\Gamma^c_{ab}v^a v^b = \mathcal{G}^{cd}\left[C^a_{db} + \mathcal{G}_{aA}C^A_{db} - \mathcal{G}_{dA}C^a_{Ab} - \mathcal{G}_{dA}\mathcal{G}_{aB}C^B_{Ab} - \mathcal{G}_{dA}\frac{\partial \mathcal{G}_{aA}}{\partial q^i}(\rho_E)^i_b\right] v^a v^b.$$

Thus, if we denote by $\tilde{C}^a_{bc}$ the real function given by

$$\tilde{C}^a_{bc} = -\mathcal{G}^{cd}\left[C^a_{db} + \mathcal{G}_{aA}C^A_{db} - \mathcal{G}_{dA}C^a_{Ab} - \mathcal{G}_{dA}\mathcal{G}_{aB}C^B_{Ab} - \mathcal{G}_{dA}\frac{\partial \mathcal{G}_{aA}}{\partial q^i}(\rho_E)^i_b\right] \tag{B.5}$$

it follows that Eqs. (B.3) may be written as follows

$$\begin{cases} \dot{v}^c = v^a v^b \tilde{C}^a_{bc} - (\rho^r_D)^i_c \frac{\partial V}{\partial q^i} \\ \dot{q}^i = (\rho^l_D)^i_a v^a. \end{cases}$$

where $(\rho^l_D)^i_a$ and $(\rho^r_D)^i_c$ are defined as in (B.4).

In what follows we are going to see how the functions $\tilde{C}^a_{bc}$, $(\rho^l_D)^i_a$ and $(\rho^r_D)^i_c$ can be interpreted as the local structure functions of an algebroid structure on $\tau_D : D \to Q$.

First, let us consider the following projectors,

$$P : D \oplus D^\perp \to D \qquad \text{and} \qquad \Pi : \tilde{D} \oplus D^\perp \to \tilde{D}$$

and the natural inclusions

$$i_D : D \to E \qquad \text{and} \qquad i_{\tilde{D}} : \tilde{D} \to E.$$

Proposition A.6. *Suppose that on the vector bundle $\tau_E : E \to Q$ we have a Lie algebroid structure $([\![\cdot,\cdot]\!]_E, \rho_E)$. Then, on the vector subbundle $\tau_D : D \to Q$ we have an algebroid structure given by the bracket*

$$B_D(\sigma,\eta) = P([\![i_D(\sigma), (i_{\tilde{D}} \circ \Pi)(\eta)]\!]_E)$$

for $\sigma, \eta \in \Gamma(\tau_D)$ and the anchor maps

$$\rho^l_D = \rho_E \circ i_D \qquad \textit{and} \qquad \rho^r_D = \rho_E \circ i_{\tilde{D}} \circ \Pi.$$

Moreover, in the local basis $\{\sigma_a, \sigma_A\}$ adapted to the decomposition $D \oplus \tilde{D}^\perp$, this algebroid $(B_D, \rho^l_D, \rho^r_D)$ has local structure functions given by (B.4) and (B.5).

Proof. In the local basis $\{\sigma_a, \sigma_A\}$ adapted to the decomposition $D \oplus \tilde{D}^\perp$ we have that

$$P(\sigma_a) = \sigma_a \qquad \text{and} \qquad P(\sigma_A) = \mathcal{G}_{cA}\sigma_c.$$

Note that $\sigma_A - \mathcal{G}_{cA}\sigma_c \in \Gamma(\tau_{D^\perp})$. Moreover, since $\sigma_a - \mathcal{G}^{ad}(\sigma_d - \mathcal{G}_{dA}\sigma_A) \in \Gamma(\tau_{D^\perp})$, we deduce that

$$\Pi(\sigma_a) = \mathcal{G}^{ad}(\sigma_d - \mathcal{G}_{dA}\sigma_A).$$

Then it is simple to prove that $P \circ \Pi_{|D} = id_D$ and from this it is obtained that the bracket and the anchor maps given above define an algebroid structure on $\tau_D : D \to Q$.

On the other hand,

$$\begin{aligned} B_D(\sigma_b, \sigma_c) \quad = \quad & (\frac{\partial \mathcal{G}^{ca}}{\partial q^i}(\rho_E)^i_b - \frac{\partial \mathcal{G}^{cd}}{\partial q^i}(\rho_E)^i_b \mathcal{G}_{dA}\mathcal{G}_{aA} - \mathcal{G}^{cd}\frac{\partial \mathcal{G}_{dA}}{\partial q^i}(\rho_E)^i_b \mathcal{G}_{cA})\sigma_a \\ & + \mathcal{G}^{cd}(C^a_{bd} + \mathcal{G}_{aA}C^A_{bd} - \mathcal{G}_{dA}C^a_{bA} - \mathcal{G}_{dA}\mathcal{G}_{aB}C^B_{bA})\sigma_a. \end{aligned}$$

Now, using that $\mathcal{G}^{cd}\mathcal{G}_{dA}\mathcal{G}_{aA} = \mathcal{G}^{ca} + \delta^c_a$, it follows that

$$\frac{\partial \mathcal{G}^{ca}}{\partial q^i} = \frac{\partial \mathcal{G}^{cd}}{\partial q^i}\mathcal{G}_{dA}\mathcal{G}_{aA} + \mathcal{G}^{cd}\frac{\partial \mathcal{G}_{dA}}{\partial q^i}\mathcal{G}_{aA} + \mathcal{G}^{cd}\mathcal{G}_{dA}\frac{\partial \mathcal{G}_{aA}}{\partial q^i}$$

which implies that (see (B.5))

$$B_D(\sigma_b, \sigma_c) = \tilde{C}^a_{bc}\sigma_a.$$

This ends the proof of the result. $\square$

Now let us define the following two maps:

$$D^l : \Gamma(\tau_D) \times \Gamma(\tau_D) \to \Gamma(\tau_D) \quad \text{such that} \quad D^l_\sigma \bar\sigma = P(\nabla^{\mathcal{G}}_\sigma \Pi \bar\sigma) \tag{B.6}$$

$$D^r : \Gamma(\tau_D) \times \Gamma(\tau_D) \to \Gamma(\tau_D) \quad \text{such that} \quad D^r_\sigma \bar\sigma = P(\nabla^{\mathcal{G}}_{\Pi\sigma} \bar\sigma). \tag{B.7}$$

Proposition A.7. *The map D^l defined in (B.6) is a ρ^l_D-connection and analogously the map D^r defined in (B.7) is a ρ^r_D-connection with ρ^l_D and ρ^r_D defined as in Proposition A.6.*

Proof. : It is sufficient to see that the maps D^l and D^r verify equation (A.1). In fact, for $\sigma, \bar\sigma \in \Gamma(\tau_D)$ and $f \in C^\infty(D)$, we have

$$\begin{aligned} D^l_{f\sigma}\bar\sigma &= P(\nabla^{\mathcal{G}}_{f\sigma}\Pi\bar\sigma) = f\,P(\nabla^{\mathcal{G}}_\sigma \Pi\bar\sigma) = f D^l_\sigma \bar\sigma, \\ D^l_\sigma f\bar\sigma &= P(\nabla^{\mathcal{G}}_\sigma \Pi(f\bar\sigma)) = P\left(f\nabla^{\mathcal{G}}_\sigma \Pi\bar\sigma + \rho_E(\sigma)(f)\Pi\bar\sigma\right) = \; f\,D^l_\sigma\bar\sigma + \rho^l_D(\sigma)(f)\bar\sigma, \end{aligned}$$

where in the last equality we use again that $P \circ \Pi|_D = id_D$.

On the other hand, it follows that

$$\begin{aligned} D^r_{f\sigma}\bar\sigma &= P(\nabla^{\mathcal{G}}_{\Pi(f\sigma)}\bar\sigma) = P(\nabla^{\mathcal{G}}_{f\Pi\sigma}\bar\sigma) = f\,P(\nabla^{\mathcal{G}}_{\Pi\sigma}\bar\sigma) = f D^r_\sigma\bar\sigma, \\ D^r_\sigma f\bar\sigma &= P(\nabla^{\mathcal{G}}_{\Pi\sigma} f\bar\sigma) = P\left(f\nabla^{\mathcal{G}}_{\Pi\sigma}\bar\sigma + \rho_E(\Pi\sigma)(f)\bar\sigma\right) = f\,D^r_\sigma\bar\sigma + \rho^r_D(\sigma)(f)\bar\sigma. \end{aligned}$$

□

Now, due to Proposition 4.1 (section 4), if we define

$$B_D(\sigma, \eta) = D^l_\sigma \eta - D^r_\eta \sigma,$$

for $\sigma, \eta \in \Gamma(\tau_D)$, then $(B_D, \rho^l_D, \rho^r_D)$ is an algebroid structure on $\tau_D : D \to Q$.

Having the definitions of the left and right connections D^l and D^r in terms of the Levi-Civita connection $\nabla^{\mathcal{G}}$, it is very easy to see that this bracket B_D coincides with the one defined in Proposition A.6.

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